EXTINCTION PROBABILITY IN A BIRTH-DEATH PROCESS WITH KILLING

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Abstract. We study birth-death processes on the non-negative integers where $\{1, 2, \ldots\}$ is an irreducible class and 0 an absorbing state, with the additional feature that a transition to state 0 may occur from any state. We give a condition for absorption (extinction) to be certain and obtain the eventual absorption probabilities when absorption is not certain. We also study the rate of convergence as $t \to \infty$ of the probability of absorption at time t, and relate it to the common rate of convergence of the transition probabilities which do not involve state 0. Finally, we derive upper and lower bounds for the probability of absorption at time t by applying a technique which involves the logarithmic norm of an appropriately defined operator.

Keywords and phrases: absorption, decay parameter, extinction time, persistence time, rate of convergence, logarithmic norm

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1 Introduction

We are concerned with a time-homogeneous, continuous-time Markov chain $\mathcal{X} \equiv \{X(t),\ t \geq 0\}$, taking values in the set $S \equiv \{0\} \cup C$, where $C \equiv \{1, 2, \ldots\}$ is an irreducible class and 0 an absorbing state. The q-matrix $Q \equiv (q_{ij},\ i, j \in S)$ of the chain is given by

$$\begin{aligned} &q_{i,i+1} = \lambda_i, \quad q_{i+1,i} = \mu_{i+1}, \quad q_{i0} = \gamma_i, \quad q_{ii} = -(\lambda_i + \mu_i + \gamma_i), \quad i > 0, \\ &q_{ij} = 0, \quad |i - j| > 1, \quad \text{and} \quad q_{0j} = 0, \quad j \geq 0, \end{aligned} \tag{1}$$

where $\lambda_i > 0$, $\mu_{i+1} > 0$ and $\gamma_i \geq 0$ for $i > 0$, and $\mu_1 = 0$. Following, for example, Karlin and Tavaré [21], we will refer to a process of this type as a *birth-death process with killing*. The parameters λ_i and μ_i are the *birth rate* and *death rate*, respectively, in state $i \in C$, while γ_i is the rate of absorption, or *killing rate*, from i into the absorbing state 0. Since, in state 1, "death" and "killing" have the same effect, the assumption $\mu_1 = 0$ is no restriction of generality. Note that Q will be conservative over C if and only if $\gamma_i = 0$ for all $i \in C$. However, we will assume in what follows that $\gamma_i > 0$ for at least one state $i \in C$, so that 0 is accessible from C. We write $\mathbb{P}_i(.) \equiv \Pr\{. \mid X(0) = i\}$.

We will assume that the process $\mathcal{X}$ is non-explosive (Q is *regular*), or, equivalently (see Chen et al. [4, Theorem 7]),

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=1}^{n} (1 + \gamma_i)\pi_i = \infty, \tag{2}$$

where

$$\pi_1 \equiv 1, \quad \pi_i \equiv \frac{\lambda_1 \lambda_2 \ldots \lambda_{i-1}}{\mu_2 \mu_3 \ldots \mu_i}, \quad i > 1. \tag{3}$$

Hence, the transition function $P(.) \equiv \{p_{ij}(.),\ i, j \in S\}$, where

$$p_{ij}(t) \equiv \mathbb{P}_i(X(t) = j), \quad i, j \in S,\ t \geq 0,$$

is the unique Q-function (transition function with q-matrix Q), is honest, and satisfies the system

$$P'(t) = QP(t) = P(t)Q, \quad t \geq 0, \tag{4}$$

of backward *and* forward equations (see, for example, Anderson [1]).

By T we denote *killing time*, that is, the (possibly defective) random variable representing the time at which absorption in state 0 occurs. In the terminology of population modelling T is the *extinction time* or *persistence time*. In what follows we shall be mainly interested in the functions

$$\tau_i(t) \equiv \mathbb{P}_i(T \leq t), \quad i \in C, \ t \geq 0,$$

and their limits

$$\tau_i \equiv \lim_{t\to\infty} \tau_i(t), \quad i \in C.$$

We will refer to $\tau_i(t)$ and τ_i as the *extinction probability at time* t and the *eventual extinction probability*, respectively, when the initial state is i. Note that $\tau_i(t) = p_{i0}(t)$.

After collecting some preliminary results in the next section we will obtain a necessary and sufficient condition for certain extinction, and an explicit expression for the eventual extinction probability in Section 3. In Section 4 we address the problem of obtaining the rate of convergence of $\tau_i(t)$ to its limit. In a pure birth-death process ($\gamma_i = 0$ for $i > 1$) this rate equals the common rate of convergence of the transition probabilities $p_{ij}(t),\ i, j \in C$, but this is not true in general in the setting at hand. We give a sufficient condition for equality of the rates of convergence. We also indicate how, if the rates are equal, results for pure birth-death processes may be invoked in the present setting. In Section 5 we derive bounds for the extinction probability $\tau_i(t)$ by applying the method developed by the second author in [27] - [29] to the model at hand, and indicate how the results may be generalized to non-homogeneous processes. We conclude with an example in Section 6.

Apart from their interest per se our results are instructive because they are indicative of the phenomena occurring once one wanders off the beaten track of the pure birth-death process.

2 Preliminaries

It is well known (see, for example, Anderson [1, Theorem 5.1.9]) that under our assumptions regarding the Markov chain $\mathcal{X}$ there exist strictly positive constants c_{ij} (with $c_{ii} = 1$) and a parameter $\alpha \geq 0$ such that

$$p_{ij}(t) \leq c_{ij} e^{-\alpha t}, \quad i, j \in C, \ t \geq 0 \tag{5}$$

and

$$\alpha = - \lim_{t \to \infty} \frac{1}{t} \log p_{ij}(t), \quad i, j \in C. \tag{6}$$

The parameter α is known as the *decay parameter* of $\mathcal{X}$ in C. It follows easily from (5) and (6) that α is also the rate of convergence to zero of the transition probabilities $p_{ij}(t)$ in the sense that

$$\alpha = \inf \left\{ x \geq 0 : \int_0^\infty e^{xt} p_{ij}(t) dt = \infty \right\}, \quad i, j \in C. \tag{7}$$

The rate of convergence of the extinction probabilities $\tau_i(t)$ to their limits τ_i will be denoted by α_0, that is,

$$\alpha_0 \equiv \inf \left\{ x \geq 0 : \int_0^\infty e^{xt} (\tau_i - \tau_i(t)) dt = \infty \right\}, \quad i \in C. \tag{8}$$

It is easily seen by an irreducibility argument that α_0 is independent of i.

The transition rates of $\mathcal{X}$ determine polynomials R_n through the recurrence relation

$$\begin{aligned} \lambda_n R_{n+1}(x) &= (\lambda_n + \mu_n + \gamma_n - x) R_n(x) - \mu_n R_{n-1}(x), \quad n > 1, \\ \lambda_1 R_2(x) &= \lambda_1 + \gamma_1 - x, \quad R_1(x) = 1. \end{aligned} \tag{9}$$

Generalizing Karlin and McGregor's [20] classic result, it is shown in [12] that the transition probabilities $p_{ij}(t), \ i, j \in C$, may be represented in the form

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} R_i(x) R_j(x) \psi(dx), \quad t \geq 0, \tag{10}$$

where ψ is a Borel measure of total mass 1 on $[0, \infty)$ with respect to which the polynomials R_n are orthogonal. (The crux of the argument in [12] is that with each q-matrix of type (1) one can associate a unique *conservative* q-matrix of type (1) such that the corresponding transition functions are *similar* in the sense of [25].) It is easy to see with [12, Theorem 4] and our Lemma 1 below that, under our assumption (2), the orthogonalizing measure for $\{R_n\}$ is in fact unique. Since the transition probabilities $p_{ij}(t), \ i, j \in C$, tend to zero as t tends to infinity (recall our assumption $\gamma_i > 0$ for at least one state i), the integral representation (10) tells us that the measure ψ cannot have a point mass at zero. It now follows readily from (7) and (10) that

$$\alpha = \min \operatorname{supp}(\psi), \tag{11}$$

which generalizes an earlier result for birth-death processes (see, for example, [11, Theorem 3.1]).

Since orthogonal polynomials have no zeros outside the support of their orthogonalizing measure, while the smallest point of the support is a limit point of zeros (see, for example Chihara [7, Section II.4]), (11) implies

$$R_n(x) > 0 \;\text{ for all }\; n \geq 1 \quad \Longleftrightarrow \quad x \leq \alpha. \tag{12}$$

It will also be useful to observe that

$$\lambda_n \pi_n \left(R_{n+1}(x) - R_n(x) \right) = \sum_{j=1}^{n} (\gamma_j - x) \pi_j R_j(x), \quad n \geq 1, \tag{13}$$

whence

$$R_n(x) = 1 + \sum_{k=1}^{n-1} \frac{1}{\lambda_k \pi_k} \sum_{j=1}^{k} (\gamma_j - x) \pi_j R_j(x), \quad n > 1. \tag{14}$$

It follows in particular that the quantities $r_n \equiv R_n(0)$ satisfy

$$r_1 = 1 \;\text{ and }\; r_n = 1 + \sum_{k=1}^{n-1} \frac{1}{\lambda_k \pi_k} \sum_{j=1}^{k} \gamma_j \pi_j r_j, \quad n > 1. \tag{15}$$

We let

$$r_\infty \equiv \lim_{n \to \infty} r_n = 1 + \sum_{k=1}^{\infty} \frac{1}{\lambda_k \pi_k} \sum_{j=1}^{k} \gamma_j \pi_j r_j, \tag{16}$$

and note the following.

Lemma 1 *We have $r_\infty = \infty$ if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k \pi_k} \sum_{j=1}^{k} \gamma_j \pi_j = \infty. \tag{17}$$

Proof The sufficiency is obvious because $r_n \geq 1$. So let us define

$$\beta_k \equiv \frac{1}{\lambda_k \pi_k} \sum_{j=1}^{k} \gamma_j \pi_j, \quad k \geq 1,$$

and assume that $\sum \beta_k$ converges. Since r_n is increasing in n we have

$$r_{n+1} = r_n + \frac{1}{\lambda_n \pi_n} \sum_{j=1}^{n} \gamma_j \pi_j r_j \leq r_n(1 + \beta_n), \quad n \geq 1,$$

so that

$$r_{n+1} \leq \prod_{k=1}^{n} (1 + \beta_k), \quad n \geq 1.$$

But $\prod(1 + \beta_k)$ and $\sum \beta_k$ converge together, so we must have $r_\infty < \infty$, as required. □

We conclude this section with representations for the extinction and eventual extinction probabilities. Indeed, the forward equations tell us that

$$p'_{i0}(t) = \sum_{j \in C} \gamma_j p_{ij}(t), \quad i \in C, \ t \geq 0.$$

It follows that

$$\tau_i(t) = p_{i0}(t) = \sum_{j \in C} \gamma_j \int_0^t p_{ij}(u) du, \quad i \in C, \ t \geq 0, \tag{18}$$

which, upon substitution of (10) and interchanging the integrals, leads to

$$\tau_i(t) = \sum_{j \in C} \gamma_j \pi_j \int_0^\infty (1 - e^{-xt}) R_i(x) R_j(x) \frac{\psi(dx)}{x}, \quad i \in C, \ t \geq 0.$$

Letting $t \to \infty$ subsequently yields

$$\tau_i = \sum_{j \in C} \gamma_j \pi_j \int_0^\infty R_i(x) R_j(x) \frac{\psi(dx)}{x}, \quad i \in C, \tag{19}$$

(by monotone convergence) and hence

$$\tau_i(t) = \tau_i - \sum_{j \in C} \gamma_j \pi_j \int_0^\infty e^{-xt} R_i(x) R_j(x) \frac{\psi(dx)}{x}, \quad i \in C, \ t \geq 0. \tag{20}$$

The expression (19) will be evaluated in the next section, and $\tau_i(t)$ will be studied in the Sections 4 and 5.

3 Eventual extinction probability

We note that by conditioning on the first event in $\mathcal{X}$ (or using the recurrence relation (9) in (19)), the eventual extinction probabilities τ_i are readily seen to satisfy the recurrence

$$(\lambda_i + \mu_i + \gamma_i)\tau_i = \lambda_i \tau_{i+1} + \mu_i \tau_{i-1} + \gamma_i, \quad i > 1,$$
$$(\lambda_1 + \gamma_1)\tau_1 = \lambda_1 \tau_2 + \gamma_1.$$

In view of (19) with $x = 0$, it follows that τ_i may be expressed in terms of τ_1 and $r_i \equiv R_i(0)$ as

$$1 - \tau_i = (1 - \tau_1) r_i, \quad i \in C. \tag{21}$$

Since $\{\tau_i,\ i \in C\}$ constitutes the smallest non-negative solution of (21) (cf. Feller [14, p. 403]) we must have $\tau_i = 1 - r_i/r_\infty$, with the interpretation that $\tau_i = 1$ whenever $r_\infty = \infty$. This result may also be obtained from Lemma 3.1 of Brockwell [3], who studies eventual extinction probabilities in a more general setting (see also Anderson [1, Section 9.2]). Considering Lemma 1 a simpler criterion for certain extinction avails us in the setting at hand. Summarizing, we conclude the following.

Theorem 2 *If (17) is satisfied then $\tau_i = 1$ for all $i \in C$, otherwise the eventual extinction probabilities satisfy*

$$\tau_i = 1 - \frac{r_i}{r_\infty} < 1, \quad i \in C, \tag{22}$$

with r_i and r_∞ given by (15) and (16), respectively.

In view of this result the condition (2) for non-explosiveness may be rephrased as follows. A necessary and sufficient condition for non-explosiveness of $\mathcal{X}$ is that either eventual extinction is certain or

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=1}^{n} \pi_i = \infty. \tag{23}$$

As might be expected, the latter is precisely the condition for non-explosiveness of $\mathcal{X}^* \equiv [\mathcal{X} \mid T = \infty]$, the (pure birth-death) process one gets by setting $\gamma_i = 0$ for all $i \in C$ (see [1, Section 8.1]).

4 Rate of convergence

In addition to accessibility of state 0 we will assume in this section that absorption at 0 is certain, that is, eventual extinction is certain and hence (17) is satisfied. Pakes [26, p. 122] has observed (see also Elmes et al. [13]) that the latter assumption is no restriction because if $\tau_i < 1$ we can work with the (Markov) process $\bar{\mathcal{X}} \equiv [\mathcal{X} \,|\, T < \infty]$, which has transition rates $\bar{q}_{ij} = q_{ij}\tau_j/\tau_i$, and transition probabilities $\bar{p}_{ij}(t) = p_{ij}(t)\tau_j/\tau_i$. Here $\tau_0 \equiv 1$, and $\tau_i > 0$ because of our accessibility assumption. It follows that

$$\bar{\tau}_i(t) \equiv \bar{p}_{i0}(t) = p_{i0}(t)/\tau_i = \tau_i(t)/\tau_i \to 1 \ \text{ as } \ t \to \infty, \quad i \in C.$$

We note from (20) that $\xi_i(t) \equiv 1 - \tau_i(t) = \mathbb{P}_i(T > t)$, the *survival probability* at time t, can be represented in the form

$$\xi_i(t) = \sum_{j\in C} \gamma_j \pi_j \int_0^\infty e^{-xt} R_i(x) R_j(x) \frac{\psi(dx)}{x}, \quad i \in C, \ t \geq 0. \tag{24}$$

In view of (11) (recall that ψ does not have an atom at 0) it is therefore tempting to believe that $\alpha_0 = \alpha$, but this is not true in general. Since $1 \geq \xi_i(t) \geq p_{ii}(t)$ we do know, however, that

$$0 \leq \alpha_0 \leq \alpha. \tag{25}$$

This was observed already by Kingman [24, Theorem 8] and more recently by Jacka and Roberts [19, (3.1.4)], whose example with strict inequalities in (25) is encompassed in the setting which is described next.

Suppose the killing rates satisfy $\gamma_i \geq \gamma > 0$ for all $i \in C$. Then we may look upon the process $\mathcal{X}$ as a birth-death process with killing $\tilde{\mathcal{X}}$, say, with rates $\tilde{\lambda}_i \equiv \lambda_i$, $\tilde{\mu}_i \equiv \mu_i$ and $\tilde{\gamma}_i \equiv \gamma_i - \gamma$, which is subject to an additional killing event taking place at rate γ. Evidently, absorption at 0 of $\mathcal{X}$ is certain. By conditioning on the time of the additional killing event we have $p_{ij}(t) = e^{-\gamma t}\tilde{p}_{ij}(t)$, $i, j \in C$, and hence

$$\alpha(\mathcal{X}) = \gamma + \alpha(\tilde{\mathcal{X}}). \tag{26}$$

By conditioning again we also obtain

$$\xi_i(t) = e^{-\gamma t}(1 - \tilde{\tau}_i(t)) = e^{-\gamma t}(1 - \tilde{\tau}_i) + e^{-\gamma t}(\tilde{\tau}_i - \tilde{\tau}_i(t)), \quad i \in C, \ t \geq 0,$$

where $\tilde{\tau}_i(t)$ is the extinction probability at time t of the process $\tilde{\mathcal{X}}$ and $\tilde{\tau}_i$ its limit as $t \to \infty$. Hence

$$\alpha_0(\mathcal{X}) = \begin{cases} \gamma & \text{if } \tilde{\tau}_1 < 1 \\ \gamma + \alpha_0(\tilde{\mathcal{X}}) & \text{if } \tilde{\tau}_1 = 1. \end{cases} \tag{27}$$

It follows that strict inequalities prevail in (25) when $\tilde{\tau}_1 < 1$ and $\alpha(\tilde{\mathcal{X}}) > 0$. We note in addition that the calculation of $\alpha_0(\mathcal{X})$ is reduced to the calculation of $\alpha_0(\tilde{\mathcal{X}})$ if $\tilde{\tau}_1 = 1$.

It has been shown in [19] (in a more general setting and implicitly assuming certain absorption) that we have $\alpha_0 = \alpha$ if only finitely many γ_i's are positive, which is also obvious from the representation (24). A more general result is the following.

Theorem 3 *If $\alpha > 0$ and eventual extinction is certain, then we have*

$$\sum_{j \in C} \gamma_j \pi_j R_j(\alpha) = \alpha \sum_{j \in C} \pi_j R_j(\alpha), \tag{28}$$

and $\alpha_0 = \alpha$ whenever either sum in (28) converges.

Proof Recalling that $R_j(\alpha) > 0$, and using an argument similar to that in the proof of [10, Theorem 4.1] it is not difficult to show with (10) that, if $\alpha > 0$,

$$q_j \equiv \lim_{t \to \infty} \frac{p_{ij}(t)}{\sum_{k \in C} p_{ik}(t)} = \frac{\pi_j R_j(\alpha)}{\sum_{k \in C} \pi_k R_k(\alpha)}, \quad j \in C, \tag{29}$$

which is to be interpreted as 0 if the sum diverges. On the other hand, since extinction is certain we have $\sum_{j \in C} p_{ij}(t) = \xi_i(t)$, and hence we may use the representation (24) to calculate q_j in a similar fashion, yielding

$$q_j = \lim_{t \to \infty} \frac{p_{ij}(t)}{\xi_i(t)} = \frac{\alpha \pi_j R_j(\alpha)}{\sum_{k \in C} \gamma_k \pi_k R_k(\alpha)}, \quad j \in C, \tag{30}$$

again with the interpretation 0 if the sum diverges. Since the two limits must be equal (28) must hold good. Moreover, if either sum in (28) converges, then $q_j > 0$ (and (29) tells us that, actually, $\{q_j,\ j \in C\}$ constitutes a proper distribution). Evidently (see also [19, Theorem 3.3.2 (*ii*)]), the latter is a sufficient condition for $\alpha_0 = \alpha$. □

Remark Theorem 3 generalizes part of the Lemma in Good [15] (see also [10, Theorem 3.2]), which concerns pure birth-death processes. When $\gamma_i > 0$ for infinitely many states i the situation differs essentially from the pure birth-death setting in that we may have $\alpha > 0$ and divergence of the series in (28) simultaneously. If either series in (28) converges then the quantities q_j of (29) (or (30)) constitute a *quasi-stationary distribution* (see, for example, Pakes [26]). In this case we also have

$$\alpha_0 = \alpha = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_i(T > t)$$

(see, [26, Lemma 2.1]).

If $\alpha_0 = \alpha$, then the problem of determining α_0 can be reduced to that of finding the decay parameter in a pure birth-death process, for which many results are available (see [5], [6], [9], [11], [16], [22], [23], [27], [28], [29]). Indeed, define $\tilde{\mathcal{X}} \equiv \{\tilde{X}(t),\ t \geq 0\}$ to be the birth-death process on C with birth and death rates

$$\tilde{\lambda}_i \equiv \lambda_i \frac{r_{i+1}}{r_i} \quad \text{and} \quad \tilde{\mu}_{i+1} \equiv \mu_{i+1} \frac{r_i}{r_{i+1}}, \quad i \in C, \tag{31}$$

respectively, where $r_i \equiv R_i(0)$. Letting $\tilde{\mu}_1 = \mu_1 = 0$, it is easy to see from (31) and (9) that

$$\tilde{\lambda}_i \tilde{\mu}_{i+1} = \lambda_i \mu_{i+1} \quad \text{and} \quad \tilde{\lambda}_i + \tilde{\mu}_i = \lambda_i + \mu_i + \gamma_i, \quad i \in C.$$

By [12, Theorem 1], this implies that there are constants $\sigma_{ij} > 0$ such that

$$p_{ij}(t) = \sigma_{ij} \tilde{p}_{ij}(t), \quad i, j \in C,\ t \geq 0,$$

with $\tilde{p}_{ij}(t)$ denoting the transition probabilities of $\tilde{\mathcal{X}}$. (In the terminology of [25] the processes $\mathcal{X}$ and $\tilde{\mathcal{X}}$ are *similar*). Consequently, $\mathcal{X}$ and $\tilde{\mathcal{X}}$ have the same decay parameter.

5 Bounds for the survival probability

To obtain bounds for $\xi_i(t) \equiv \mathbb{P}_i(T > t)$, the survival probability at time t, we choose the approach used in [27] - [29] for pure birth-death processes (see also [17] or [18] for an exposition of the method). Application of the technique requires the elements of the q-matrix Q to be bounded, so in what follows we assume that

$$\sup_i \{\lambda_i + \mu_i + \gamma_i\} < \infty.$$

We let $A \equiv (q_{ij},\ i,j \in C)$, the matrix that remains after removing the first row and column from Q, and define

$$\boldsymbol{x}_i(t) \equiv (p_{i1}(t), p_{i2}(t), \ldots)^T, \quad i \in C,\ t \geq 0,$$

where superscript ^{T} denotes transpose. Further, let $D \equiv \text{diag}(d_1, d_2, \ldots)$, with $d_1, d_2, \ldots$ denoting positive parameters, and $\boldsymbol{z}_i(t) \equiv D\boldsymbol{x}_i(t)$. The forward equations for $P(.)$ then tell us that

$$\boldsymbol{z}_i'(t) = DAD^{-1}\boldsymbol{z}_i(t), \quad i \in C,\ t \geq 0.$$

If the parameters d_i are such that DAD^{-1} can be interpreted as a bounded linear operator on a normed space, then the theory expounded, for example, in [29] and [17] reveals that for all $i \in C$ and $t \geq 0$

$$\exp\{-t\theta^*(\boldsymbol{d})\}\, ||\boldsymbol{z}_i(0)|| \leq ||\boldsymbol{z}_i(t)|| \leq \exp\left\{tg(DAD^{-1})\right\} ||\boldsymbol{z}_i(0)||, \tag{32}$$

where

$$\theta^*(\boldsymbol{d}) \equiv \sup_{i \in C} \left\{ \lambda_i + \mu_i + \gamma_i - \lambda_i \frac{d_{i+1}}{d_i} - \mu_i \frac{d_{i-1}}{d_i} \right\}, \tag{33}$$

with $\boldsymbol{d} \equiv (d_1, d_2, \ldots)$ and $d_0 \equiv 0$, and

$$g(DAD^{-1}) \equiv \lim_{h \downarrow 0} \frac{||I + hDAD^{-1}|| - 1}{h},$$

the *logarithmic norm* of the operator DAD^{-1}. Moreover, choosing $||.|| = ||.||_1$, the ℓ_1-norm, we have

$$-g(DAD^{-1}) = \theta(\boldsymbol{d}) \equiv \inf_{i \in C} \left\{ \lambda_i + \mu_i + \gamma_i - \lambda_i \frac{d_{i+1}}{d_i} - \mu_i \frac{d_{i-1}}{d_i} \right\}. \tag{34}$$

Hence (32) translates into

$$d_i e^{-\theta^* t} \leq \sum_{j \in C} d_j p_{ij}(t) \leq d_i e^{-\theta t}, \quad i \in C,\ t \geq 0, \tag{35}$$

where $\theta \equiv \theta(\boldsymbol{d})$ and $\theta^* \equiv \theta^*(\boldsymbol{d})$. As an aside we note that $\theta(\boldsymbol{d}) = \theta^*(\boldsymbol{d}) = x$ if and only if $d_i = cR_i(x)$ for some constant c, as can easily be seen from the recurrence relation (9). It follows in particular that

$$\sum_{j \in C} R_j(x) p_{ij}(t) = R_i(x) e^{-xt}, \quad i \in C,\ t \geq 0, \tag{36}$$

from which the representation (10) may be derived (cf. Karlin and McGregor [20, Section I.2]).

Since

$$\xi_i(t) \equiv \mathbb{P}_i(T > t) = \sum_{j \in C} p_{ij}(t),$$

the inequalities (35) immediately give us the following bounds for the extinction probability $\xi_i(t)$.

Theorem 4 *(i) Let $d_j \geq 1$ for all $j \in C$ and $\theta \equiv \theta(\boldsymbol{d})$ as in (34), then*

$$\xi_i(t) \leq d_i e^{-\theta t}, \quad i \in C,\ t \geq 0. \tag{37}$$

(ii) Let $d_j \leq 1$ for all $j \in C$ and $\theta^* \equiv \theta^*(\boldsymbol{d})$ as in (33), then

$$\xi_i(t) \geq d_i e^{-\theta^* t}, \quad i \in C,\ t \geq 0. \tag{38}$$

Note that eventual extinction must be certain when $d_j \geq 1$ for all $j \geq 1$ and $\theta(\boldsymbol{d}) > 0$.

Corollary 5 *If the constants $\mu \geq 0$ and $a \geq 0$ are such that*

$$\mu < \mu_{j+1} \quad \textit{and} \quad a \leq \mu + \gamma_j - \frac{\lambda_j \mu}{\mu_{j+1} - \mu}, \quad j = 1, 2, \ldots,$$

then

$$\xi_i(t) \leq e^{-at} \prod_{j=1}^{i} \frac{\mu_{j+1}}{\mu_{j+1} - \mu}, \quad i \in C,\ t \geq 0. \tag{39}$$

Proof Choosing $d_1 = 1$ and $d_{j+1}/d_j = \mu_{j+1}/(\mu_{j+1} - \mu)$ for $j \geq 1$, we have $d_j \geq 1$ and

$$\theta(\boldsymbol{d}) = \inf_{j \in C} \left\{ \mu + \gamma_j - \frac{\lambda_j \mu}{\mu_{j+1} - \mu} \right\},$$

so that the conditions of Theorem 4 *(i)* are satisfied. Substitution in (37) gives the result. □

Taking $\mu = 0$ it follows in particular that $\xi_i(t) \le e^{-at}$ if $a \le \inf\{\gamma_j\}$, as we had observed already by a different argument in the previous section.

If α, the decay parameter of $\mathcal{X}$ in C, is known, then the following corollary might be useful. Recall that $R_j(\alpha) > 0$ by (12).

Corollary 6 *If* $0 \le R_{min} < R_j(\alpha) < R_{max} \le \infty$ *for all* j *then*

$$\frac{R_i(\alpha)}{R_{max}} e^{-\alpha t} < \xi_i(t) < \frac{R_i(\alpha)}{R_{min}} e^{-\alpha t}, \quad i \in C, \ t \ge 0, \tag{40}$$

where the left-hand (right-hand) side should be interpreted as zero (infinity) if $R_{max} = \infty$ *(*$R_{min} = 0$*).*

Proof We have noticed already that letting $d_j = cR_j(x)$ for some constant c gives us $\theta(\boldsymbol{d}) = \theta^*(\boldsymbol{d}) = x$. Hence, if $R_j(\alpha) > R_{min} > 0$ for all j, then the conditions of Theorem 4 *(i)* are satisfied if we choose $a = \alpha$ and $d_j = R_j(\alpha)/R_{min}$, and substitution in (37) gives the upper bound. On the other hand, if $R_j(\alpha) < R_{max} < \infty$ for all j, then the conditions of Theorem 4 *(ii)* are satisfied if we choose $a = \alpha$ and $d_j = R_j(\alpha)/R_{max}$, and substitution in (38) gives the lower bound. □

Under certain circumstances (35) may lead to other bounds for $\xi_i(t)$. For example, suppose that $\gamma_i > 0$ for all $i \in C$, and choose $d_i = \gamma_i$ in (34) – (35), so that

$$\gamma_i e^{-\theta^* t} \le \sum_{j \in C} \gamma_j p_{ij}(t) \le \gamma_i e^{-\theta t}, \quad i \in C, \ t \ge 0,$$

where $\theta \equiv \theta(\boldsymbol{\gamma})$, $\theta^* \equiv \theta^*(\boldsymbol{\gamma})$ and $\boldsymbol{\gamma} \equiv (\gamma_1, \gamma_2, \ldots)$. If $\theta > 0$ we obtain, in view of (18), for $\xi_i(t) \equiv 1 - \tau_i(t)$ the bounds

$$1 - \frac{\gamma_i}{\theta}\left(1 - e^{-\theta t}\right) \le \xi_i(t) \le 1 - \frac{\gamma_i}{\theta^*}\left(1 - e^{-\theta^* t}\right), \quad i \in C, \ t \ge 0. \tag{41}$$

At the other extreme end, suppose that $\gamma_i = 0$ for $i > 1$, that is, we are dealing with a pure birth-death process. Now choose $d_i \le d_1$ for all i in (34) – (35), and suppose $\theta \equiv \theta(\boldsymbol{d}) > 0$. Then we have

$$\gamma_1 p_{i1}(t) \le \frac{\gamma_1}{d_1} \sum_{j \in C} d_j p_{ij}(t) \le \gamma_1 \frac{d_i}{d_1} e^{-\theta t}, \quad i \in C, \ t \ge 0,$$

by (35), and hence, by (18) again,

$$\xi_i(t) \geq 1 - \frac{\gamma_1}{\theta}\frac{d_i}{d_1}\left(1 - e^{-\theta t}\right), \quad i \in C,\ t \geq 0. \tag{42}$$

We conclude this section by noting that the result (35) can easily be generalized to non-homogeneous processes. Specifically, let $\mathcal{X}$ be birth-death processes with killing with time-dependent birth rates $\lambda_n(t)$, death rates $\mu_n(t)$, and killing rates $\gamma_n(t)$. Then, under appropriate boundedness conditions and for all $i \in C$ and $t \geq 0$,

$$d_i \exp\left\{-\int_0^t \theta^*(u)du\right\} \leq \sum_{j\in C} d_j p_{ij}(t) \leq d_i \exp\left\{-\int_0^t \theta(u)du\right\}, \tag{43}$$

where

$$\theta(\boldsymbol{d}, t) \equiv \inf_{i\in C}\left\{\lambda_i(t) + \mu_i(t) + \gamma_i(t) - \lambda_i(t)\frac{d_{i+1}}{d_i} - \mu_i(t)\frac{d_{i-1}}{d_i}\right\},\ t \geq 0, \tag{44}$$

and

$$\theta^*(\boldsymbol{d}, t) \equiv \sup_{i\in C}\left\{\lambda_i(t) + \mu_i(t) + \gamma_i(t) - \lambda_i(t)\frac{d_{i+1}}{d_i} - \mu_i(t)\frac{d_{i-1}}{d_i}\right\},\ t \geq 0. \tag{45}$$

The corresponding generalisations of Theorem 4 and Corollary 5 are straightforward.

6 Example

Interesting cases arise if $\gamma_i > 0$ for infinitely many states i, while γ_i is not constant for all i. We will analyse a simple example satisfying these conditions, namely the process with transition rates

$$\lambda_i \equiv \lambda,\ \mu_i \equiv \mu\mathbb{I}_{\{i>1\}} \text{ and } \gamma_i \equiv \gamma\mathbb{I}_{\{i>1\}}, \quad i \in C, \tag{46}$$

for some constants $\lambda > 0$, $\mu > 0$ and $\gamma > 0$, where $\mathbb{I}_E$ denotes the indicator function of an event E. It is easily seen that (17) is satisfied so that extinction is certain. The polynomials R_n of (9) satisfy the recurrence relation

$$\begin{aligned} &\lambda R_{n+1}(x) = (\lambda + \mu + \gamma - x)R_n(x) - \mu R_{n-1}(x), \quad n > 1, \\ &\lambda R_2(x) = \lambda - x, \quad R_1(x) = 1, \end{aligned} \tag{47}$$

which, by the transformation

$$S_n(x) \equiv (-1)^n \left(\frac{\lambda}{\mu}\right)^{n/2} R_{n+1}(\lambda+\mu+\gamma+2x\sqrt{\lambda\mu}), \quad n \geq 0, \tag{48}$$

reduces to

$$\begin{aligned} &S_n(x) = 2xS_{n-1}(x) - S_{n-2}(x), \quad n > 1, \\ &S_1(x) = 2x + \eta, \quad S_0(x) = 1, \end{aligned} \tag{49}$$

where

$$\eta \equiv \frac{\mu+\gamma}{\sqrt{\lambda\mu}}. \tag{50}$$

The polynomials S_n can be represented as

$$S_n(x) = U_n(x) + \eta U_{n-1}(x), \quad n \geq 1, \tag{51}$$

where $U_n(x)$ denote the *Chebysev polynomials of the second kind*. The latter satisfy the recurrence

$$\begin{aligned} &U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n > 1, \\ &U_1(x) = 2x, \quad U_0(x) = 1, \end{aligned} \tag{52}$$

and may be represented as

$$U_n(x) = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}}, \quad x = \frac{1}{2}(z + z^{-1}), \quad n \geq 0. \tag{53}$$

It will be useful to observe that

$$U_n(x) = (-1)^n U_n(-x) \text{ and } U_n(1) = n+1. \tag{54}$$

By appropriately transforming the orthogonalizing measure for $\{S_n(x)\}$ given in Chihara [7, p. 205] we can conclude that the polynomials R_n are orthogonal with respect to a measure which consists of a positive density on the interval

$$\left(\lambda+\mu+\gamma-2\sqrt{\lambda\mu},\, \lambda+\mu+\gamma+2\sqrt{\lambda\mu}\right),$$

and, if $\mu+\gamma > \sqrt{\lambda\mu}$, an atom at the point $\lambda\gamma/(\mu+\gamma)$. Since

$$\frac{\lambda\gamma}{\mu+\gamma} = \lambda+\mu+\gamma-\sqrt{\lambda\mu}\left(\eta+\eta^{-1}\right), \tag{55}$$

it thus follows from (11) that

$$\alpha = \lambda+\mu+\gamma - \begin{cases} 2\sqrt{\lambda\mu} & \text{if} \quad \mu+\gamma \leq \sqrt{\lambda\mu} \\ \sqrt{\lambda\mu}\left(\eta+\eta^{-1}\right) & \text{if} \quad \mu+\gamma \geq \sqrt{\lambda\mu}. \end{cases} \tag{56}$$

We next wish to determine the value of α_0. To this end we will not try to employ (24), but rather argue as follows. Let E_a denote an exponentially distributed random variable with mean a^{-1}, and B a random variable representing the busy period in an $M/M/1$ queueing system with arrival rate λ and service rate μ. (If $\lambda > \mu$ the distribution of B is defective.) A little reflection then shows that, if the initial state is 1, the extinction time T may be represented as

$$T = E_\lambda + E_\gamma \mathbb{I}_{\{E_\gamma \leq B\}} + (B + T^*)\mathbb{I}_{\{E_\gamma > B\}},$$

where T and T^* are independent but identically distributed. It follows that

$$\begin{aligned}\tilde{\tau}(s) \equiv \mathbb{E}[e^{-sT}] &= \mathbb{E}\left[e^{-sT}\mathbb{I}_{\{E_\gamma \leq B\}} + e^{-sT}\mathbb{I}_{\{E_\gamma > B\}}\right] \\ &= \mathbb{E}\left[e^{-sE_\lambda}\left(e^{-sE_\gamma}\mathbb{I}_{\{E_\gamma \leq B\}} + e^{-s(B+T^*)}\mathbb{I}_{\{E_\gamma > B\}}\right)\right] \\ &= \frac{\lambda}{\lambda + s}\left(\mathbb{E}\left[e^{-sE_\gamma}\mathbb{I}_{\{E_\gamma \leq B\}}\right] + \tilde{\tau}(s)\mathbb{E}\left[e^{-sB}\mathbb{I}_{\{E_\gamma > B\}}\right]\right),\end{aligned}$$

so that

$$\left(\lambda + s - \lambda\mathbb{E}\left[e^{-sB}\mathbb{I}_{\{E_\gamma > B\}}\right]\right)\tilde{\tau}(s) = \lambda\mathbb{E}\left[e^{-sE_\gamma}\mathbb{I}_{\{E_\gamma \leq B\}}\right]. \tag{57}$$

A little algebra reveals that

$$\mathbb{E}\left[e^{-sE_\gamma}\mathbb{I}_{\{E_\gamma \leq B\}}\right] = \frac{\gamma}{\gamma + s}(1 - \tilde{B}(\gamma + s)),$$

and

$$\mathbb{E}\left[e^{-sB}\mathbb{I}_{\{E_\gamma > B\}}\right] = \tilde{B}(\gamma + s),$$

where $\tilde{B}(s) \equiv \mathbb{E}[e^{-sB}]$. Substitution of these results in (57) gives us

$$\tilde{\tau}(s) = \frac{\gamma(\lambda - \lambda\tilde{B}(\gamma + s))}{(\gamma + s)(\lambda + s - \lambda\tilde{B}(\gamma + s))}. \tag{58}$$

It is well known (see, for instance, Cohen [8, Eq. (II.2.31)]) that

$$\tilde{B}(s) = \frac{1}{2\lambda}\left(\lambda + \mu + s - \sqrt{(\lambda + \mu + s)^2 - 4\lambda\mu}\right),$$

which, upon substitution in (58) and some algebra, leads to

$$\tilde{\tau}(s) = \frac{\gamma\left(s^2 + (\lambda + \mu + \gamma)s + 2\lambda\gamma - s\sqrt{(\lambda + \mu + \gamma + s)^2 - 4\lambda\mu}\right)}{2(\gamma + s)(\lambda\gamma + (\mu + \gamma)s)}. \tag{59}$$

By inverting this expression we can obtain an explicit formula for $\tau_1(t)$, the extinction time distribution when the initial state is 1. At this point, however, we are interested only in α_0 – the rate of convergence of $\tau_1(t)$ – which, apart from a minus sign, equals the singularity of $\tilde{\tau}(s)$ which is closest to the imaginary axis. Since the largest branch point at $-\gamma-(\sqrt{\lambda}-\sqrt{\mu})^2$ is always smaller than the pole at $-\gamma$ it follows that $\alpha_0=\gamma$ or $\alpha_0=\lambda\gamma/(\mu+\gamma)$, depending on whether $\lambda \geq \mu+\gamma$ or $\lambda \leq \mu+\gamma$, respectively.

Collecting all our results we conclude the following.

Theorem 7 *The process with transition rates (46) has rates of convergence* α_0 *and* α *given by*

$$\alpha_0=\alpha=\frac{\lambda\gamma}{\mu+\gamma} \quad \text{if} \quad \lambda \leq \mu+\gamma,$$

$$\alpha_0=\gamma<\alpha=\frac{\lambda\gamma}{\mu+\gamma} \quad \text{if} \quad \sqrt{\lambda\mu} \leq \mu+\gamma<\lambda,$$

and

$$\alpha_0=\gamma<\alpha=\gamma+\left(\sqrt{\lambda}-\sqrt{\mu}\right)^2 \quad \text{if} \quad \mu+\gamma<\sqrt{\lambda\mu}.$$

Observe that our findings are in accordance with the intuitive result that α_0 must tend to zero as γ tends to zero.

It is interesting to establish how much of the information in Theorem 7 may be obtained from Theorem 3. To this end we note that, by (3) and (46),

$$\pi_{n+1}=\left(\frac{\lambda}{\mu}\right)^n, \quad n \geq 0, \tag{60}$$

so that, by (48),

$$\pi_{n+1}R_{n+1}(x)=(-1)^n\left(\frac{\lambda}{\mu}\right)^{n/2} S_n\left(\frac{x-\lambda-\mu-\gamma}{2\sqrt{\lambda\mu}}\right), \quad n \geq 0. \tag{61}$$

Hence, it follows after some algebra from (51), (53) and (54) that, for $n \geq 0$,

$$\pi_{n+1}R_{n+1}(\alpha)=\begin{cases} (1+(1-\eta)n)\left(\dfrac{\lambda}{\mu}\right)^{n/2} & \text{if} \quad \mu+\gamma \leq \sqrt{\lambda\mu} \\ \left(\dfrac{\lambda}{\mu+\gamma}\right)^n & \text{if} \quad \mu+\gamma \geq \sqrt{\lambda\mu}. \end{cases} \tag{62}$$

Since $\sqrt{\lambda\mu}<\mu+\gamma$ if $\lambda<\mu+\gamma$, while $\lambda>\mu$ if $\lambda \geq \mu+\gamma$, we conclude that the series in (28) converge if and only if $\lambda<\mu+\gamma$. Hence, Theorem 3 tells us that $\alpha_0=\alpha$ if $\lambda<\mu+\gamma$. In the opposite case Theorem 3 does not help us.

By extending the method by which we have calculated $\tilde{\tau}(s)$ we can obtain the Laplace-Stieltjes transform of the extinction time distribution when the initial state is any state $i \in C$ rather than 1. By inversion we can therefore, in principle at least, calculate $\tau_i(t)$, and hence $\xi_i(t)$. But the procedure is cumbersome so it is of interest to apply the methodology of Section 5 to the present example. For instance, choosing $d_1 = 1$ and

$$d_{j+1} = \left(\frac{\mu}{\mu+\gamma}\right)^j, \quad j \geq 1,$$

in (33) gives us $\theta^* = \lambda\gamma/(\mu+\gamma)$ and hence, by Theorem 4 *(ii)*,

$$\xi_i(t) \geq \left(\frac{\mu}{\mu+\gamma}\right)^{i-1} \exp\left\{\frac{-\lambda\gamma t}{\mu+\gamma}\right\}, \quad i \in C, \ t \geq 0. \tag{63}$$

This is also the bound produced by Corollary 6 when $\mu+\gamma \geq \sqrt{\lambda\mu}$. In the case $\mu+\gamma < \sqrt{\lambda\mu}$ Corollary 6 yields a lower bound which we will not spell out, but improves upon (63) for t sufficiently large.

As an aside we finally note that ours is yet another example, next to the examples in Pakes [26] and Bobrowski [2], showing that *asymptotic remoteness*, that is,

$$\lim_{i\to\infty} p_{i0}(t) = 0, \quad t \geq 0, \tag{64}$$

is not necessary for the existence of a quasi-stationary distribution. Indeed, it is obvious that (64) is not satisfied in the present setting, while, in view of (62) and the Remark following Theorem 3, a quasi-stationary distribution does exist when $\lambda < \mu+\gamma$.

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