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# WEAK ERGODICITY OF $M_t/M_t/N/N+R$ QUEUE

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We consider nonstationary birth and death processes on finite state space and study the bounds of the rate of convergence to the limit regime. We also obtain some bounds on the rate of convergence for the queue-length process of $M_t/M_t/N/N+R$ queue.

## 1. Introduction

Consider a queueing model with $N$ servers and $R \geq 0$ waiting rooms. Let $X(t)$ be a number of customers in the queue. Then $X(t)$ is a birth and death process (BDP) with state space $E_{N+R} = \{0, 1, \ldots, N+R\}$ and birth and death rates $a_n(t) = \lambda(t)$, $b_n(t) = \mu(t) \min(n, N)$ respectively. The most known model corresponds to the case $R = 0$, this is the famous Erlang loss system. General approach for the study of nonstationary BDPs has been proposed in our papers [5, 6], see also [1, 2, 3, 7]. Namely we study the forward Kolmogorov system and special transformations of intensity matrices.

In this note we outline our general approach for the study of such models (in Section 2) and obtain some new bounds on the rate of convergence for queue-length process of $M_t/M_t/N/N+R$ queue (Section 3).

Let $X(t)$ be a BDP on finite state space $\{0, \ldots, S\}$ and let $\lambda_n(t), \mu_{n+1}(t)$, $n = 0, \ldots, S-1$ be the respective birth and death intensities. We assume that all

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$\lambda_n(t), \mu_{n+1}(t)$, $n = 0, \dots, S-1$, are non-negative and locally integrable on $[0; \infty)$ functions.

Denote by $p_i(t)$ the state probabilities of $X(t)$, by $\mathbf{p}(t) = (p_0(t), p_1(t), \dots, p_S(t))^T$ the respective column vector, and by $A(t) = \{a_{ij}(t)\}_{i,j=0}^{S}$, $t \geq 0$ the transposed intensity matrix of the process:

$$a_{ij}(t) = \begin{cases} \lambda_{i-1}(t), & \text{if} \quad j = i-1, \\ \mu_{i+1}(t), & \text{if} \quad j = i+1, \\ -(\lambda_i(t) + \mu_i(t)), & \text{if} \quad j = i, \\ 0, & \text{overwise}. \end{cases} \tag{1}$$

Throughout the whole paper we use the $l_1$-norm for vectors $\|\mathbf{x}\| = \sum |x_i|$.

BDP $X(t)$ is called weakly ergodic if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(s), \mathbf{p}^{**}(s)$ and any $s \geq 0$.

## 2. General bounds for finite BDPs

Let $\delta_k > 0$, $1 \leq k \leq S-1$ be positive numbers.

Put

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) - \delta_{k+1}\lambda_{k+1}(t) - \delta_k^{-1}\mu_k(t), \quad k = 0, \dots, S-1, \tag{2}$$

and

$$\zeta_k(t) = \lambda_k(t) + \mu_{k+1}(t) + \delta_{k+1}\lambda_{k+1}(t) + \delta_k^{-1}\mu_k(t), \quad k = 0, \dots, S-1, \tag{3}$$

(here we suppose $\delta_0^{-1} = \delta_S = 0$ and $\delta_0 = 1$).

Denote

$$\min_{0 \leq k \leq S-1} \alpha_k(t) = \beta(t), \tag{4}$$

$$\max_{0 \leq k \leq S-1} \zeta_k(t) = \chi(t), \tag{5}$$

and

$$d_k = \prod_{i=0}^{k-1} \delta_i, \quad \theta = \sum_{i=1}^{S} d_i, \quad d = \min_{1 \leq i \leq S} d_i \tag{6}$$

**Theorem 1.** *The following bounds on the rate of convergence hold:*

$$\frac{d}{4\theta}e^{-\int_s^t \chi(\tau)\,d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\| \le \|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\| \le \frac{4\theta}{d}e^{-\int_s^t \beta(\tau)\,d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|, \tag{7}$$

*for any initial probability distributions* $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$, *and any* $0 \le s \le t$.

Proof. We use the approach of [6]. Consider the forward Kolmogorov system for the probabilistic dynamics of the process:

$$\frac{d\mathbf{p}}{dt} = A(t)\,\mathbf{p}, \quad t \ge 0. \tag{8}$$

By introducing

$$p_0(t) = 1 - \sum_{i\ge 1} p_i(t),$$

we obtain from (8) the following system:

$$\frac{d\mathbf{z}}{dt} = B(t)\,\mathbf{z} + \mathbf{f}(t), \tag{9}$$

where

$$B = \begin{pmatrix} -(\lambda_0+\lambda_1+\mu_1) & \mu_2-\lambda_0 & -\lambda_0 & \cdots & -\lambda_0 & -\lambda_0 \\ \lambda_1 & -(\lambda_2+\mu_2) & \mu_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(\lambda_{S-1}+\mu_{S-1}) & \mu_S \\ 0 & 0 & 0 & \cdots & \lambda_{S-1} & -\mu_S \end{pmatrix}, \tag{10}$$

$$\mathbf{z} = (p_1,\dots,p_S)^T, \ \mathbf{f}(t) = (a_{10}(t),\dots,a_{S0}(t))^T = (\lambda_0(t),0,\dots,0)^T. \tag{11}$$

Consider the matrix

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots & d_1 \\ 0 & d_2 & d_2 & \cdots & d_2 \\ 0 & 0 & d_3 & \cdots & d_3 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & 0 & d_S \end{pmatrix}, \tag{12}$$

and the respective vector norm $\|\mathbf{z}\|_{1D} = \|D\mathbf{z}\|$.

Let $\mathbf{x}(t) = (x_1, x_2, ..., x_S)$. Then

$$\|\mathbf{x}\|_{1D} = \|D\mathbf{x}\| \leq \|D\|\|\mathbf{x}\| = \theta\|\mathbf{x}\|; \tag{13}$$

and

$$\|\mathbf{x}\| = \|D^{-1}D\mathbf{x}\| \leq \|D^{-1}\|\|\mathbf{x}\|_{1D} \leq \frac{2}{d}\|\mathbf{x}\|_{1D} \tag{14}$$

so

$$\frac{d}{2}\|\mathbf{x}\| \leq \|\mathbf{x}\|_{1D} \leq \theta\|\mathbf{x}\|. \tag{15}$$

Consider the logarithmic norm of $B(t)$ in $l_{1D}$-norm:

$$\begin{aligned} \gamma(B(t))_{1D} &= \gamma(DB(t)D^{-1}) \\ &= \max_{0\leq i\leq S-1} \left(-\lambda_i(t) - \mu_{i+1}(t) + \delta_{i+1}\lambda_{i+1}(t) + \delta_i^{-1}\mu_i(t)\right) \\ &= -\min_{0\leq i\leq S-1} \left(\lambda_i(t) + \mu_{i+1}(t) - \delta_{i+1}\lambda_{i+1}(t) - \delta_i^{-1}\mu_i(t)\right) \\ &= -\beta(t). \end{aligned} \tag{16}$$

and the logarithmic norm of $-B(t)$ in $l_{1D}$-norm:

$$\begin{aligned} &\gamma(-B(t))_{1D} \\ &\quad= \max_{0\leq i\leq S-1} \left(\lambda_i(t) + \mu_{i+1}(t) + \delta_{i+1}\lambda_{i+1}(t) + \delta_i^{-1}\mu_i(t)\right) = \chi(t). \end{aligned} \tag{17}$$

Then the following bound holds:

$$e^{-\int_s^t \chi(\tau)d\tau} \leq \|V(t,s)\|_{1D} \leq e^{-\int_s^t \beta(\tau)d\tau}, \tag{18}$$

for any $t, s \quad (0 \leq s \leq t)$, where $V(t,s)$ is the Cauchy operator of equation (9).

Now we have from (15) and (18)

$$\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\| \le 2\|\mathbf{z}^*(t)-\mathbf{z}^{**}(t)\| \le \frac{4}{d}\|\mathbf{z}^*(t)-\mathbf{z}^{**}(t)\|_{1D}$$

$$\le \frac{4}{d}e^{-\int\limits_s^t \beta(\tau)\,d\tau}\|\mathbf{z}^*(s)-\mathbf{z}^{**}(s)\|_{1D}$$

$$\le \frac{4}{d}e^{-\int\limits_s^t \beta(\tau)\,d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|_{1D}$$

$$\le \frac{4\theta}{d}e^{-\int\limits_s^t \beta(\tau)\,d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|. \tag{19}$$

On the other hand, for any solution $\mathbf{y_1}, \mathbf{y_2}$ of the system (9) one has:

$$\|\mathbf{y_1}(t)-\mathbf{y_2}(t)\|_{1D} \ge e^{-\int_s^t \chi(\tau)d\tau}\|\mathbf{y_1}(s)-\mathbf{y_2}(s)\|_{1D}. \tag{20}$$

Now let $\mathbf{y_1}, \mathbf{y_2}$ be such that

$$\mathbf{p}^*(s) = \left(1-\|\mathbf{y_1}(s)\|, \quad \mathbf{y_1^T}(s)\right)^T; \quad \mathbf{p}^{**}(s) = \left(1-\|\mathbf{y_2}(s)\|, \quad \mathbf{y_2^T}(s)\right)^T. \tag{21}$$

Hence (14) and (20) imply the following bound:

$$\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\| \ge \frac{1}{\theta}\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\|_{1D} = \frac{1}{\theta}\|\mathbf{y_1}(t)-\mathbf{y_2}(t)\|_{1D} \ge$$

$$\frac{1}{\theta}e^{-\int_s^t \chi(\tau)d\tau}\|\mathbf{y_1}(s)-\mathbf{y_2}(s)\|_{1D} =$$

$$\frac{1}{2\theta}e^{-\int_s^t \chi(\tau)d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|_{1D} \ge \frac{d}{4\theta}e^{-\int_s^t \chi(\tau)d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|. \tag{22}$$

Note that the equality $\int\limits_0^\infty \beta(\tau)\,d\tau = +\infty$ implies weak ergodicity of $X(t)$.

**Remark.** Under the assumptions of Theorem 1 the following estimate holds

$$e^{-\int_s^t \chi(\tau)\,d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|_{1D} \tag{23}$$

$$\le \|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\|_{1D} \le e^{-\int_s^t \beta(\tau)\,d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|_{1D},$$

for any initial probability distributions $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$, and any $0 \le s \le t$.

Put

$$\beta^*(t) = \max_{0 \le k \le S-1} \alpha_k(t). \tag{24}$$

**Theorem 2.** *Let $D\left(\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\right) \ge \mathbf{0}$ (or $D\left(\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\right) \le \mathbf{0}$). Then the following bound holds:*

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \ge \frac{d}{2\theta} e^{-\int\limits_s^t \beta^*(u)du} \left\|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\right\|, \tag{25}$$

*for any $s \ge 0$, $t \ge s$.*

Proof. Consider the system (9) and put $\mathbf{v}(t) = D\left(\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\right)^T$. Then

$$\frac{d\mathbf{v}(t)}{dt} = DB(t)D^{-1}\mathbf{v}(t). \tag{26}$$

Let now $l_1^+$ be a set of $l_1$-vectors with non-negative coordinates. All non-diagonal elements of the matrix $DB(t)D^{-1}$ are non-negative for any $t \ge 0$. Therefore, if $\mathbf{v}(s) \ge 0$ for some $s \ge 0$ then $\mathbf{v}(t) \ge 0$ for any $t \ge s$.

Therefore

$$\frac{d\left\{\sum v_i\right\}}{dt} \ge \left\{\sum v_i\right\} \min_k \left(-\lambda_k - \mu_{k+1} + \delta_{k+1}\lambda_{k+1} + \delta_k^{-1}\mu_k\right) =$$

$$-\left\{\sum v_i\right\} \max_k(\lambda_k + \mu_{k+1} - \delta_{k+1}\lambda_k - \delta_k^{-1}\mu_k) = -\beta^* \left\{\sum v_i\right\}. \tag{27}$$

Hence we have for any $t \ge s$ :

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \ge \frac{1}{\theta}\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} = \frac{\|\mathbf{v}(t)\|}{\theta} = \frac{\sum v_i(t)}{\theta} \ge$$

$$\frac{1}{\theta}e^{-\int_s^t \beta^*(\tau)d\tau} \sum v_i(s) = \frac{1}{\theta}e^{-\int_s^t \beta^*(\tau)d\tau}\|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D} \ge$$

$$\frac{d}{2\theta}e^{-\int_s^t \beta^*(\tau)d\tau}\|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|. \tag{28}$$

This inequality implies our claim. □

**Remark.** Under the assumptions of Theorem 2 the following bound holds

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \geq e^{-\int_s^t \beta^*(\tau)\,d\tau} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}, \tag{29}$$

for any initial probability distributions $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$ such that

$$D\left(\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\right) \geq \mathbf{0} \text{ (or } D\left(\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\right) \leq \mathbf{0}),$$

and for any $0 \leq s \leq t$.

**Remark.** One can see that the main problem now is the finding of the appropriate sequence $\{\delta_k\}$.

## 3. Bounds for the queue-length process of $M_t/M_t/N/N+R$ queue.

Let now $X(t)$ be queue-length process for $M_t/M_t/N/N+R$ queue. In the case $R = 0$ (Erlang model with losses) we have necessary and sufficient condition of weak ergodicity in the following form (see the proof and related bounds in [4, 8]):

The process is weakly ergodic if and only if

$$\int_0^\infty \left(\lambda(t) + \mu(t)\right)\,dt = +\infty. \tag{30}$$

Here we consider general case $R > 0$. Then we have in (2) and (3)

$$\alpha_k(t) = \lambda(t) + (k+1)\mu(t) - \delta_{k+1}\lambda(t) - \delta_k^{-1}k\mu(t),$$

if $0 \leq k \leq N-1$,

$$\alpha_k(t) = \lambda(t) + N\mu(t) - \delta_{k+1}\lambda(t) - \delta_k^{-1}N\mu(t)$$

if $N \leq k < N+R$,
and

$$\zeta_k(t) = \lambda(t) + (k+1)\mu(t) + \delta_{k+1}\lambda(t) + \delta_k^{-1}k\mu(t)$$

if $0 \le k \le N-1$,

$$\zeta_k(t) = \lambda(t) + N\mu(t) + \delta_{k+1}\lambda(t) + \delta_k^{-1}N\mu(t),$$

if $N \le k < N+R$, respectively.

**First case.** Let there exist $l > 1$ such that the following assumption holds:

$$\int\limits_0^{\infty} \left(N\mu(\tau) - l\lambda(\tau)\right)\, d\tau = +\infty. \tag{31}$$

Put $\delta_k = 1,\ k \le N-1$, and $\delta_k = l,\ k \ge N$.
Then

$$\alpha_k(t) = \begin{cases} \mu(t), & k < N-1; \\ \mu(t) - (l-1)\lambda(t), & k = N-1; \\ \left(1 - \frac{1}{l}\right)\left(N\mu(t) - l\lambda(t)\right), & N \le k \le N+R-2; \\ N\mu(t)\left(1 - \frac{1}{l}\right) + \lambda(t), & k = N+R-1. \end{cases} \tag{32}$$

We can suppose $l \le \dfrac{N}{N-1}$, hence

$$\beta(t) = \min_k \alpha_k(t) = \left(1 - \frac{1}{l}\right)\left(N\mu(t) - l\lambda(t)\right), \tag{33}$$

$$\beta^*(t) = \max_k \alpha_k(t) = \mu(t), \tag{34}$$

and

$$\chi(t) = \max_k \zeta_k(t) \le 2\left(l\lambda(t) + N\mu(t)\right). \tag{35}$$

Therefore Theorems 1 and 2 imply the following statement.

**Theorem 3.** *Let* (31) *be fulfilled. Then the following bounds hold:*

$$\frac{1}{4\theta}e^{-2\int_s^t(l\lambda(\tau)+N\mu(\tau))\,d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\| \le \|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\| \le$$

$$4\theta e^{-\int_s^t\left(1-\frac{1}{l}\right)(N\mu(\tau)-l\lambda(\tau))\,d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|, \tag{36}$$

*for any initial probability distributions* $\mathbf{p}^*(s),\ \mathbf{p}^{**}(s)$, *and any* $0\le s\le t$,

$$\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\| \ge \frac{1}{2\theta}e^{-\int\limits_s^t\mu(u)du}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|, \tag{37}$$

*if* $D(\mathbf{p}^*(s)-\mathbf{p}^{**}(s))\ge\mathbf{0}$ *(or* $D(\mathbf{p}^*(s)-\mathbf{p}^{**}(s))\le\mathbf{0}$*), and any* $0\le s\le t$, *where* $\theta=N-1+\sum\limits_{i=1}^{R+1}l^i$.

**Second case.** Let there exist $l<1$ such that

$$\int\limits_0^\infty(l\lambda(\tau)-N\mu(\tau))\,d\tau=+\infty. \tag{38}$$

Put $\delta_k=l,\ k\ge1$. Then

$$\alpha_k(t)=\begin{cases}\left(\frac{1}{l}-1\right)(l\lambda(t)-k\mu(t))+\mu(t), & k\le N-1;\\ \left(\frac{1}{l}-1\right)(l\lambda(t)-N\mu(t)), & N\le k\le N+R-2;\\ \lambda(t)-N\left(\frac{1}{l}-1\right)\mu(t), & k=N+R-1\end{cases} \tag{39}$$

and

$$\beta(t)=\min_k\alpha_k(t)=\left(\frac{1}{l}-1\right)(l\lambda(t)-N\mu(t)), \tag{40}$$

$$\beta^*(t) = \max_k \alpha_k(t) \le \lambda(t). \tag{41}$$

On the other hand,

$$\chi^*(t) = \max_k \zeta_k(t) \le 2\left(\lambda(t) + \frac{N}{l}\mu(t)\right). \tag{42}$$

Hence Theorems 1 and 2 imply the following statement.

**Theorem 4.** *Let* (38) *be fulfilled. Then the following bounds hold:*

$$\frac{d}{4\theta}e^{-2\int_s^t\left(\lambda(\tau)+\frac{N}{l}\mu(\tau)\right)d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\| \le \|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\| \le$$

$$\frac{4\theta}{d}e^{-\int_s^t\left(\frac{1}{l}-1\right)(l\lambda(\tau)-N\mu(\tau))\,d\tau}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|, \tag{43}$$

*for any initial probability distributions* $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$, *and any* $0 \le s \le t$,

$$\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\| \ge \frac{d}{2\theta}e^{-\int\limits_s^t \lambda(u)du}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|, \tag{44}$$

*if* $D(\mathbf{p}^*(s)-\mathbf{p}^{**}(s)) \ge \mathbf{0}$ *(or* $D(\mathbf{p}^*(s)-\mathbf{p}^{**}(s)) \le \mathbf{0}$*), and any* $0 \le s \le t$, *where* $\theta \le N+R$ *and* $d = l^{N+R}$.

**Remark.** There is a number of open problems for $M_t/M_t/N/N+R$ queue, see [9, 10]. For instance, the condition (30) seems to be necessary and sufficient for weak ergodicity of the queue-length process. Is this true?

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## REFERENCES

[1] B. L. Granovsky, A. I. Zeifman. The decay function of nonhomogeneous birth-death processes, with application to mean-field models. *Stoch. Proc. Appl.*, **72** (1997), 105–120.

[2] B. L. Granovsky, A. I. Zeifman. The $N$-limit of spectral gap of a class of birth-death Markov chains. *Appl. Stoch. Models in Business and Industry*, **16** (2000), 235–248.

[3] B. L. Granovsky, A. I. Zeifman. Nonstationary Queues: Estimation of the Rate of Convergence. *Queueing Systems*, **46** (2004), 363–388.

[4] A. I. Zeifman. Properties of a system with losses in the case of variable rates. *Automat. Remote Control*, **50** (1989), 82-–87.

[5] A. I. Zeifman. Some estimates of the rate of convergence for birth and death processes. *J. Appl. Probab.*, **28** (1991), 268–277.

[6] A. I. Zeifman. Upper and lower bounds on the rate of convergence for nonhomogeneous birth and death processes. *Stoch. Proc. Appl.*, **59** (1995), 157–173.

[7] A. I. Zeifman, S. Leorato, E. Orsingher, Ya. Satin, G. Shilova. Some universal limits for nonhomogeneous birth and death processes. *Queueing systems*, **52** (2006), 139–151.

[8] A. I. Zeifman. On nonstationary Erlang model. *Automation and Remote Control*, **70** (2009), 2003–2012.

[9] A. I. Zeifman. Ergodicity of $M_t/M_t/N/N+R$ queue and related bounds. Submitted to *Queueing Systems*. ????

[10] A. I. Zeifman. Ergodicity of finite birth-death processes and related bounds. Submitted to *Communications in Statistics – Theory and Methods*. ????

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