# On $M_n(t)/M_n(t)/S$ queues with catastrophes


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## ABSTRACT

We consider the $M_n(t)/M_n(t)/S$ queue with catastrophes. The bounds on the rate of convergence to the limit regime and the estimates of the limit probabilities are obtained.



## Keywords

General nonstationary Markovian queues, catastrophes, weak ergodicity


## 1. INTRODUCTION

The simplest (stationary) queueing models with catastrophes have been studied some years ago, see for instance [1–6, 9]. Namely, when the queue is not empty, catastrophes may occur with the respective rates. The effect of each catastrophe is to make the queue instantly empty. Simultaneously, the system becomes ready to accept new customers. Nonstationary Markovian queueing models (birth-death processes) with catastrophes have been studied in our previous papers [11, 13] in the case if catastrophes rates do not depend on the length of queue. It is extremely difficult to obtain general results for arbitrary forms of the birth, death and catastrophe intensities and therefore we must content ourselves with obtaining various types of approximations. Here we consider general Markovian model of $M_n(t)/M_n(t)/S$ queue with catastrophes and suppose that the catastrophes rates depend on the length of queue. Let $X = X(t)$, $t \geq 0$ be a queue-length process for this model. Then $X = X(t)$ is a birth and death process (BDP) with catastrophes and birth, death, and catastrophe rates $\lambda_n(t) = \nu_n \lambda(t)$, $\mu_n(t) = \eta_n \mu(t)$ and $\xi_n(t) = \zeta_n \xi(t)$ respectively.

Let $p_{ij}(s,t) = Pr\{X(t) = j\,|X(s) = i\}$ for $i, j \geq 0$, $0 \leq s \leq t$ be the transition probability functions of the process $X = X(t)$ and $p_i(t) = Pr\{X(t) = i\}$ be the state probabilities.

The probabilistic dynamics of the process is represented by the forward Kolmogorov system of differential equations:

$$\begin{cases} \frac{dp_0}{dt} = -\lambda_0(t)p_0 + \mu_1(t)p_1 + \sum_{k\geq 1} \xi_k(t)p_k, \\ \frac{dp_k}{dt} = \lambda_{k-1}(t)p_{k-1} - (\lambda_k(t) + \mu_k(t) + \xi_k(t))\,p_k + \\ \qquad \mu_{k+1}(t)p_{k+1}, k \geq 1. \end{cases} \quad (1)$$

We denote by $\mathbf{p}(t) = (p_0(t), p_1(t), \dots)^T$, $t > 0$ the column vector of state probabilities and by $A(t) = \{a_{ij}(t),\ t \geq 0\}$ the matrix related to (1). One can see that $A(t) = Q^T(t)$, where $Q(t)$ is the intensity (or infinitesimal) matrix for $X(t)$.

We assume that all basic arrival, service and catastrophe rates $\lambda(t)$, $\mu(t)$ and $\xi(t)$ are locally integrable on $[0,\infty)$ functions. Moreover, we suppose that $0 \leq \nu_n + \eta_n + \zeta_n \leq M$, hence we can rewrite the system (1) in the form

$$\frac{d\mathbf{p}}{dt} = A(t)\,\mathbf{p}, \quad \mathbf{p} = \mathbf{p}(t), \quad t \geq 0, \quad (2)$$

as a differential equation in the space of sequences $l_1$ with bounded operator function $A(t)$. Therefore we can apply the general approach to employ the logarithmic norm of a matrix for the study of the problem of stability of Kolmogorov system of differential equations associated with nonhomogeneous Markov chains. The method is based on the following two components: the logarithmic norm of a linear operator and a special similarity transformation of the matrix of intensities of the Markov chain considered, see the respective definitions, bounds, references and other details in [10–13].

## 2. WEAK ERGODICITY AND RELATED BOUNDS

We first consider some definitions.

DEFINITION 1. *A Markov chain* $X(t)$ *is called weakly ergodic, if* $\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\| \to 0$ *as* $t\to\infty$ *for any initial conditions* $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$*, where* $\|\mathbf{x}\|$ *is* $l_1$*-norm.*

Here $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the respective solutions of (2) and $\|\mathbf{x}\|_{l_1} = \|\mathbf{x}\|_1 = \sum_i |x_i|$.

Put $E_k(t) = E\{X(t)\,|X(0)=k\}$ ( then the respective initial condition of system (2) is the $k-th$ unit vector $\mathbf{e_k}$).

DEFINITION 2. *Let* $X(t)$ *be a Markov chain. Then* $\varphi(t)$ *is called the* limiting mean *of* $X(t)$ *if*

$$\lim_{t\to\infty}(\varphi(t)-E_k(t))=0$$

*for any* $k$*.*

We study the ergodic properties of the queue-length process for the following important situations:

(i) essential catastrophe rates for any queue length;

(ii) sufficiently large service rates;

(iii) large arrival rates, and essential rates of catastrophes if the length of queue is proportional to a positive integer number.

In all of these cases we obtain weak ergodicity of $X(t)$ with sufficiently sharp explicit bounds on the speed of convergence and an existence of the limiting mean.

The results of next theorems are formulated in terms of the auxiliary sequences $\{d_i\}$, which do not possess any probabilistic meaning. A detailed analysis of their properties is given in [7], see also [8]. We note that they are a sort of counterpart of the Lyapunov functions.

Let $\{d_i\},\ i\ge 1,\ d_{-1}=d_0=1$, be a sequence of positive numbers. Put

$$\alpha_k(t) = \lambda_k(t)+\mu_{k+1}(t)+\xi_{k+1}(t) - \frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t), \quad k\ge 0, \tag{3}$$

and

$$\alpha(t) = \inf_{k\ge 0}\alpha_k(t). \tag{4}$$

Firstly we consider the following general statement.

THEOREM 1. *Let a process with rates* $\lambda_k(t)$*,* $\mu_k(t)$*, and* $\xi_k(t)$ *be given. Let us assume that there exists a sequence* $\{d_i\}$ *such that* $d=\inf_{i\ge 1} d_i>0$*, and*

$$\int_0^\infty \alpha(t)\,dt = +\infty. \tag{5}$$

*Then* $X(t)$ *is weakly ergodic, and the following bound holds:*

$$\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\|_1 \le \frac{4}{d} e^{-\int_0^t \alpha(\tau)\,d\tau} \sum_{i\ge 1} g_i |p_i^*(0)-p_i^{**}(0)|, \tag{6}$$

*for any* $t\ge 0$*, where* $g_i=\sum_{k\ge 0}^{i-1} d_k$ .

**Proof.** The property $\sum_{i=0}^\infty p_i(t)=1$ for any $t\ge s$ allows to put $p_0(t)=1-\sum_{i\ge 1}p_i(t)$ (for ordinary BDP see this way of study, for instance, in [7, 10]), then we obtain the following system from (2)

$$\frac{d\mathbf{z}(t)}{dt} = B(t)\mathbf{z}(t)+\mathbf{f}(t), \tag{7}$$

where $\mathbf{z(t)} = (p_1(t), p_2(t),\dots)^T$ , $\mathbf{f(t)} = (\lambda_0(t),0,0,\dots)^T$, $B(t) = (b_{ij}(t))_{i,j=1}^\infty$ and

$$b_{ij} = \begin{cases} -(\lambda_0+\lambda_1+\mu_1+\xi_1), & \text{if} \quad i=j=1,\\ \mu_2-\lambda_0, & \text{if} \quad i=1, j=2,\\ -\lambda_0, & \text{if} \quad i=1, j>2,\\ -(\lambda_j+\mu_j+\xi_j), & \text{if} \quad i=j>1,\\ \mu_j, & \text{if} \quad i=j-1>1,\\ \lambda_j, & \text{if} \quad i=j+1>1,\\ 0, & \text{otherwise}. \end{cases} \tag{8}$$

This is a linear non-homogeneous differential system the solution of which can be written as

$$\mathbf{z}(t) \quad = \quad V(t,0)\mathbf{z}(0) + \int_0^t V(t,\tau)\mathbf{f}(\tau)\,d\tau, \tag{9}$$

where $V(t,z)$ is the Cauchy operator of (7), see [8]. Recall that the Cauchy operator is defined as follows:

$$V(t,s) = I + \int_s^t B(t_1)\ dt_1 + \int_s^t B(t_1) \int_s^{t_1} B(t_2)\ dt_2 dt_1 + \dots,$$

and $V(t,s) = e^{(t-s)B}$ for the case of stationary process $(B(t)\equiv B)$.

Consider the matrix

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{10}$$

and the space of sequences

$$\ell_{1D} = \left\{ \mathbf{z} = (p_1, p_2, \ldots) : \|\mathbf{z}\|_{1D} = \|D\mathbf{z}\|_1 < \infty \right\}, \qquad (11)$$

as in [10], where $d_i$ are some positive numbers.

We have

$$D^{-1} = \begin{pmatrix} d_1^{-1} & -d_2^{-1} & 0 & \ddots & \\ 0 & d_2^{-1} & -d_3^{-1} & 0 & \ddots \\ \ddots & 0 & \ddots\, d_3^{-1} & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Applying this transformation to the matrix $B(t)$ in (7) , we arrive to the matrix $DB(t)D^{-1} = \left(b^1_{ij}(t)\right)_{i,j=1}^{\infty}$, where

$$b^1_{ij} = \begin{cases} -(\lambda_{j-1} + \mu_j + \xi_j), & \text{if} \quad i = j, \\ \frac{d_{i-1}}{d_i}\mu_i, & \text{if} \quad i = j-1, \\ \frac{d_{j-1}}{d_j}\lambda_j, & \text{if} \quad i = j+1, \\ 0, & \text{otherwise} . \end{cases} \qquad (12)$$

Now we can study BDP with catastrophes using the logarithmic norm and related bounds, see definitions and detail discussion in [7,8,10].

We recall that the logarithmic norm of operator function is defined as the following limit:

$$\gamma(B) = \lim_{h \to +0} \frac{\|I + B(t)\,h\| - 1}{h},$$

particularly, $\gamma(B)_1 = \sup_i \left( b_{ii} + \sum_{j \neq i} |b_{ji}| \right)$. The respective bound for Cauchy operator $V(t,s)$ holds:

$$\|V(t,s)\| \le e^{\int_s^t \gamma(\tau)\, d\tau},$$

see for instance [8,10].

We have now the following bound of the logarithmic norm $\gamma(B(t))$ in $l_{1D}$:

$$\gamma(B)_{1D} = \gamma\left(DB(t)D^{-1}\right)_1 = \sup_{i \ge 0} \left( \frac{d_{i+1}}{d_i}\lambda_{i+1}(t) + \frac{d_{i-1}}{d_i}\mu_i(t) - (\lambda_i(t) + \mu_{i+1}(t) + \xi_{i+1}(t)) \right) = \sup\left(-\alpha_k(t)\right) = -\alpha(t), \qquad (13)$$

in accordance with (3). Hence

$$\|V(t,s)\|_{1D} \le e^{\int_s^t \gamma(\tau)\, d\tau}.$$

Therefore the following inequality holds:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \le e^{-\int_s^t \alpha(\tau)\, d\tau} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}. \qquad (14)$$

Consider $\ell_1$ and $\ell_{1D}$ norms of a vector $\mathbf{z} = (z_1, z_2, \ldots)^T$, then

$$d\|\mathbf{z}\|_1 \le \sum_{i \ge 1} d_i z_i = d_1 \left( \left| \sum_{i \ge 1} z_i + \sum_{i \ge 2} -z_i \right| \right) + d_2 \left( \left| \sum_{i \ge 2} z_i + \sum_{i \ge 3} -z_i \right| \right) + \cdots \le d_1 \left| \sum_{i \ge 1} z_i \right| + 2d_2 \left| \sum_{i \ge 2} z_i \right| + \cdots \le 2\|\mathbf{z}\|_{1D}. \qquad (15)$$

On the other hand, $\|\mathbf{p}^* - \mathbf{p}^{**}\|_1 \le 2\|\mathbf{z}\|_1$ for any $\mathbf{p}^*, \mathbf{p}^{**}$ and corresponding $\mathbf{z}$. Thereby we prove our claim is true.

Corollary 1. *Let, in addition, the numbers $d_i$ grow sufficiently fast so that $\inf_{k \ge 1} \frac{d_k}{k} = \omega > 0$. Then $X(t)$ has the limiting mean, say $\phi(t)$, and the following bound holds:*

$$|\phi(t) - E_k(t)| \le \frac{4}{d\omega} e^{-\int_0^t \alpha(\tau)\, d\tau} \|\mathbf{p}(0) - \mathbf{e_k}\|_{1D}. \qquad (16)$$

Moreover, Theorem 1 gives us the explicit bounds for the limiting regime and limiting mean of $X(t)$.

Theorem 2. *Let the assumptions of Theorem 1 be satisfied. Then the following bounds hold:*

$$\liminf_{t \to \infty} \Pr\left(X(t) < k\right) \ge 1 - \frac{d_1 \nu_0}{\sum_{j=1}^k d_j} \limsup_{t \to \infty} \int_0^t \lambda(u) e^{-\int_u^t \alpha(\tau)\, d\tau}\, du, \qquad (17)$$

*and*

$$\limsup_{t \to \infty} E_{\mathbf{p}}(t) \le \frac{\nu_0}{W} \limsup_{t \to \infty} \int_0^t \lambda(u) e^{-\int_u^t \alpha(\tau)\, d\tau}\, du, \qquad (18)$$

*where $W = \inf_{k \ge 1} \frac{\sum_{i=1}^k d_i}{k}$ .*

**Proof.**

We have for any solution of (7) in $1D$-norm

$$\|\mathbf{z}(t)\| = \|V(t,0)\mathbf{z}(0) + \int_0^t V(t,z)\mathbf{f}(z)\, dz\| \le e^{-\int_0^t \alpha(\tau)\, d\tau} \|\mathbf{z}(0)\| + d_1 \nu_0 \int_0^t \lambda(u) e^{-\int_u^t \alpha(\tau)\, d\tau}\, du, \qquad (19)$$

hence

$$\limsup_{t\to\infty}\|\mathbf{z}(t)\|_{1D} \leq$$

$$d_1\nu_0 \limsup_{t\to\infty}\int_0^t \lambda(u)e^{-\int_u^t \alpha(\tau)\,d\tau}\,du. \qquad (20)$$

We consider only nonnegative solutions of (7), then

$$\|\mathbf{z}\|_{1D} = \|D\mathbf{z}\| = d_1 p_1 + (d_1+d_2)p_2 + \dots, \qquad (21)$$

and therefore

$$\sum_{i\geq k} p_i \leq \frac{1}{\sum_{j=1}^k d_j}\left(\sum_{j=1}^k d_j p_k + \dots\right) \leq \frac{1}{\sum_{j=1}^k d_j}\|\mathbf{z}\|_{1D}, \qquad (22)$$

now

$$\limsup_{t\to\infty}\sum_{i\geq k} p_k(t) \leq \qquad (23)$$

$$\frac{d_1\nu_0}{\sum_{j=1}^k d_j}\limsup_{t\to\infty}\int_0^t \lambda(u)e^{-\int_u^t \alpha(\tau)\,d\tau}\,du,$$

and we obtain (17).

Finally, bound (18) follows from the inequality

$$\sum_{k\geq 1} k p_k \leq \frac{1}{W}\sum_{k\geq 1}\sum_{j=1}^k d_j p_k = \frac{1}{W}\|\mathbf{z}\|_{1D}. \qquad (24)$$

THEOREM 3. *Let under assumptions of the previous Corollary all intensities be 1-periodic functions of $t$. Then there exists 1-periodic limiting regime, say $\pi(t) = (\pi_0(t), \pi_1(t), \dots)^T$, and the respective 1-periodic limiting mean $\phi(t) = \sum_k k\pi_k(t)$. Moreover, the following bounds hold:*

$$\|\mathbf{p}(t) - \pi(t)\|_1 \leq 4e^{-\int\limits_0^t \alpha(\tau)\,d\tau}\|\mathbf{p}(0) - \pi(0)\|_{1D}, \qquad (25)$$

$$|\phi(t) - E_k(t)| \leq \frac{4}{d\omega}e^{-\int\limits_0^t \alpha(\tau)\,d\tau}\|\pi(0) - \mathbf{e_k}\|_{1D}. \qquad (26)$$

## 3. APPLICATIONS OF THEOREM 1

Consider firstly the situation (i): essential catastrophe rates for any queue length.

THEOREM 4. *Let the intensities of birth, death and catastrophes be such that the sequence $\{\eta_n\}$ is increasing, the sequence $\{\nu_n\}$ is decreasing, $\inf_n \zeta_n = \zeta > 0$ and moreover, let there exist $\varepsilon > 0$ such that*

$$\int_0^\infty (\zeta\xi(t) - \varepsilon\nu_0\lambda(t))\,dt = +\infty. \qquad (27)$$

*Then queue-length process $X(t)$ is weakly ergodic and has the limiting mean. Moreover, choosing the limiting regime $\pi(t)$ and limiting mean $\phi(t)$, corresponding to the initial condition $X(0) = 0$, we have the following bounds:*

$$\|\mathbf{p}(t) - \pi(t)\|_1 \leq 4(1+\varepsilon)^k \varepsilon^{-1} e^{-\int\limits_0^t (\zeta\xi(\tau) - \varepsilon\nu_0\lambda(\tau))\,d\tau}, \qquad (28)$$

*and*

$$|E_k(t) - E_0(t)| \leq \frac{4(1+\varepsilon)^k}{\varepsilon\omega} e^{-\int\limits_0^t (\zeta\xi(\tau) - \varepsilon\nu_0\lambda(\tau))\,d\tau}, \qquad (29)$$

*for any $X(0) = k$.*

**Proof.** Put $d_{-1} = d_0 = 1$, $d_{k+1} = (1+\varepsilon)d_k$, $k \geq 0$. Then we obtain the following bound:

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) + \xi_{k+1}(t) - \frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t) \geq \qquad (30)$$

$$(\nu_k - (1+\varepsilon)\nu_{k+1})\lambda(t) + \left(\eta_{k+1} - \frac{1}{1+\varepsilon}\eta_k\right)\mu(t) + \zeta\xi(t) \geq \zeta\xi(t) - \varepsilon\nu_k\lambda(t).$$

Then our claim follows from (6) and (16).

Situation (ii): sufficiently large service rates.

THEOREM 5. *Let the intensities of birth, death and catastrophes be such that the sequence $\{\eta_n\}$ is increasing, the sequence $\{\nu_n\}$ is decreasing and let moreover, there exist $\varepsilon > 0$ such that*

$$\int_0^\infty (\eta_1\mu(t) - (1+\varepsilon)\nu_0\lambda(t))\,dt = +\infty. \qquad (31)$$

*Then queue-length process $X(t)$ is weakly ergodic and has the limiting mean. Moreover, choosing the limiting regime $\pi(t)$ and limiting mean $\phi(t)$, corresponding to the initial condition $X(0) = 0$, we have the following bounds:*

$$\|\mathbf{p}(t) - \pi(t)\|_1 \leq 4(1+\varepsilon)^k \varepsilon^{-1} e^{-\frac{\varepsilon}{1+\varepsilon}\int\limits_0^t (\eta_1\mu(\tau) - (1+\varepsilon)\nu_0\lambda(\tau))\,d\tau}, \qquad (32)$$

*and*

$$|E_k(t) - E_0(t)| \leq \frac{4(1+\varepsilon)^k}{\varepsilon\omega} e^{-\frac{\varepsilon}{1+\varepsilon}\int\limits_0^t (\eta_1\mu(\tau) - (1+\varepsilon)\nu_0\lambda(\tau))\,d\tau}, \qquad (33)$$

*for any $X(0) = k$:*

**Proof.** Put again $d_{-1} = d_0 = 1$, $d_{k+1} = (1+\varepsilon)d_k$, $k \geq 0$. Then we have the following bound:

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) + \xi_{k+1}(t)$$
$$-\frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t) \geq \quad (34)$$
$$(\nu_k - (1+\varepsilon)\nu_{k+1})\lambda(t) + \left(\eta_{k+1} - \frac{1}{1+\varepsilon}\eta_k\right)\mu(t) \geq$$
$$\frac{\varepsilon}{1+\varepsilon}\left(\eta_k\mu(t) - (1+\varepsilon)\nu_k\lambda(t)\right).$$

Then our bounds follows from (6) and (16).

Situation (iii): large arrival rates, and essential rates of catastrophes if the length of queue is proportional to a positive integer number.

THEOREM 6. *Let the intensities of birth, death and catastrophes be such that the sequence* $\{\eta_n\} \to \eta_\infty$ *is increasing, the sequence* $\{\nu_n\} \to \nu_\infty$ *is decreasing and let moreover, there exist* $\delta > 0$*, a natural number* $N$ *such that* $\inf_n \zeta_{nN} = \zeta > 0$ *and*

$$\int_0^\infty g(t)\,dt = +\infty, \quad (35)$$

*where*

$$g(t) = \min\left(\zeta\xi(t) - \delta\nu_0\lambda(t) - \delta\eta_\infty\mu(t)\right.,$$
$$\left.\nu_\infty\lambda(t) - (1+\delta)\eta_1\mu(t)\right). \quad (36)$$

*Then queue-length process* $X(t)$ *is weakly ergodic and has the limiting mean.*

**Proof.**

Put now $d_{-1} = d_0 = 1$, $d_{k+1} = (1+\varepsilon)^{-1}d_k$, for $k \neq iN-1$, and $d_{k+1} = (1+\varepsilon)^N d_k$ if $k = iN-1$, where $\varepsilon < \delta$ is a positive number such that $(1+\varepsilon)^N - 1 < \delta$.

We have for $k \neq iN-1$ the following estimates:

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) + \xi_{k+1}(t) -$$
$$\frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t) \geq \quad (37)$$
$$\left(\nu_k - \frac{d_{k+1}}{d_k}\nu_{k+1}\right)\lambda(t) + \left(\eta_{k+1} - \frac{d_{k-1}}{d_k}\eta_k\right)\mu(t) \geq$$
$$\frac{\varepsilon}{1+\varepsilon}\left(\nu_\infty\lambda(t) - (1+\varepsilon)\eta_1\mu(t)\right).$$

For $k = iN-1$ we obtain

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) + \xi_{k+1}(t) -$$
$$\frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t) \geq \left(\nu_k - \nu_{k+1}(1+\varepsilon)^N\right)\lambda(t) +$$
$$(\eta_{k+1} - (1+\varepsilon)\eta_k)\mu(t) + \zeta\xi(t) \geq \quad (38)$$
$$\zeta\xi(t) - \varepsilon\eta_\infty\mu(t) - \left((1+\varepsilon)^N - 1\right)\nu_0\lambda(t).$$

Finally our claim follows from Theorem 1.

COROLLARY 2. *Let the assumptions of Theorem 6 be fulfilled. Then, choosing the limiting regime* $\pi(t)$ *and limiting mean* $\phi(t)$*, corresponding to the initial condition* $X(0) = 0$*, we have the following bounds:*

$$\|\mathbf{p}(t) - \pi(t)\|_1 \leq \frac{4g_k}{d} e^{-\int_s^t G(\tau)\,d\tau}, \quad (39)$$

*and*

$$|E_k(t) - E_0(t)| \leq \frac{4g_k}{d\omega} e^{-\int_0^t G(\tau)\,d\tau}, \quad (40)$$

*for any* $X(0) = k$ *, where* $d = \min d_k = (1+\varepsilon)^{-(N-1)}$*,* $g_k = \sum_{i\leq k-1} d_i$*, and*

$$G(t) = \min\left(\frac{\varepsilon}{1+\varepsilon}\left(\nu_\infty\lambda(t) - (1+\varepsilon)\eta_1\mu(t)\right),\right.$$
$$\left.\zeta\xi(t) - \varepsilon\eta_\infty\mu(t) - \left((1+\varepsilon)^N - 1\right)\nu_0\lambda(t)\right).$$

## 4. EXAMPLE

Consider the queuing model with $S$ servers, impatient customers and catastrophes. Let $X(t)$ be the respective queue-length process. Then $X(t)$ is a birth-death-catastrophes process with arrival rates $\lambda_n(t) = \frac{\lambda(t)}{\min(n,S)}$; service rates $\mu_n(t) = \min(n,S)\mu(t)$ and catastrophes rates $\xi_n(t)$, and we can apply Theorems 4-6 for the study of this process in different situations.

(i). Let $S = 100$, $\lambda(t) = 240 + \cos 2\pi t$, $\mu(t) = 1 + \sin 4\pi t$, $\xi_n(t) = 100 + \sin 4\pi t, n \geq 1$ be arrival, service and catastrophe rates rates respectively. Then the assumptions of Theorem 4 are fulfilled for $\varepsilon = 0.4$, we can apply the approach of [11] and find the limit characteristics approximately with error $10^{-5}$ as the respective characteristics of truncated process with $n = 100$ and $t \in [8.0, 9.0]$. The corresponding graphs are shown in Figures 1-3.

(ii). Let $S = 100$, $\lambda(t) = 240 + \cos 2\pi t$, $\mu(t) = 10 + \sin 4\pi t$, $\xi_n(t) = \frac{2+\sin 4\pi t}{n}$. Then the assumptions of Theorem 5 are fulfilled for $\varepsilon = 3.0$, and we find the limit characteristics approximately with error $10^{-5}$ as the respective characteristics of truncated process with $n = 65$ and $t \in [1.5, 2.5]$. The corresponding graphs are shown in Figures 4-6.

3. Let $S = 5$ $\lambda(t) = 240 + \cos 2\pi t$, $\mu(t) = 1 + \sin 4\pi t$,

$\xi_n(t) = \begin{cases} 0, & \text{if} \quad n \neq 3k, \\ 153 + \sin 4\pi t, & \text{if} \quad n = 3k \end{cases}$, then the assumptions of Theorem 6 are fulfilled for $\varepsilon = 0.15$, and we find the limit characteristics approximately with error $10^{-4}$ as the respective characteristics of truncated process with $n = 250$ and $t \in [6.0, 7.0]$. The corresponding graphs are shown in Figures 7-9.

One can compare the behavior of the limiting characteristics of the queue-length process in these cases.

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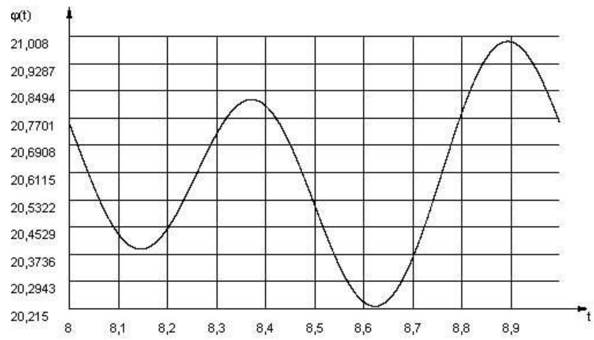


**Figure 1: Situation (i), approximation of the limiting mean**

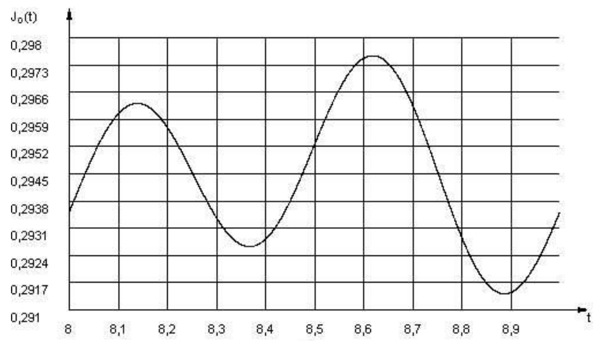


**Figure 2: Situation (i), approximation of the limit behavior of $J_0(t) = \Pr(X(t) = 0)$**

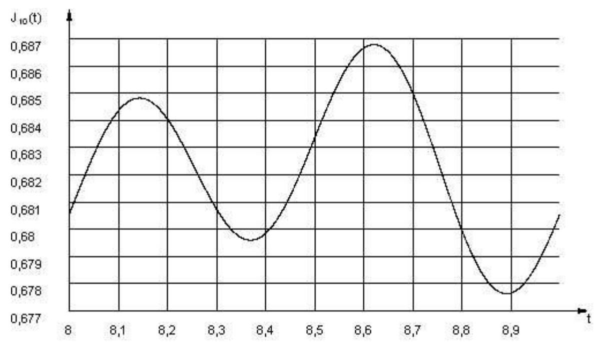


**Figure 3: Situation (i), approximation of the limit behavior of $J_{10}(t) = \Pr(X(t) \leq 10)$**

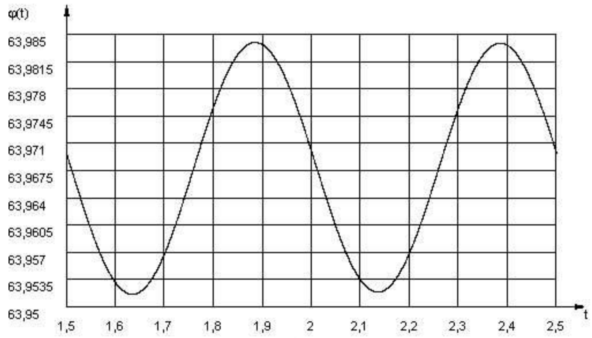


**Figure 4: Situation (ii), approximation of the limiting mean**

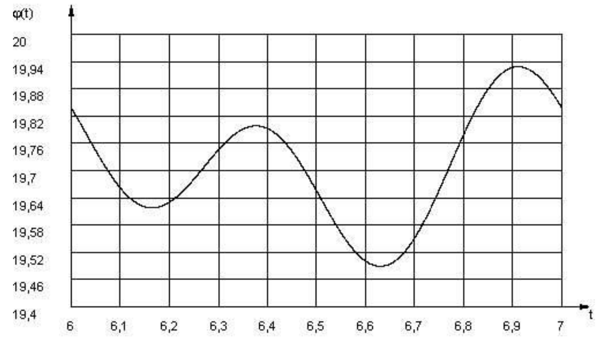


**Figure 7: Situation (iii), approximation of the limiting mean**

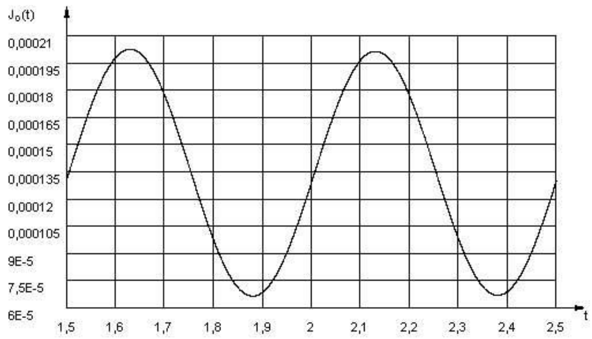


**Figure 5: Situation (ii), approximation of the limit behavior of** $J_0(t) = \Pr(X(t) = 0)$

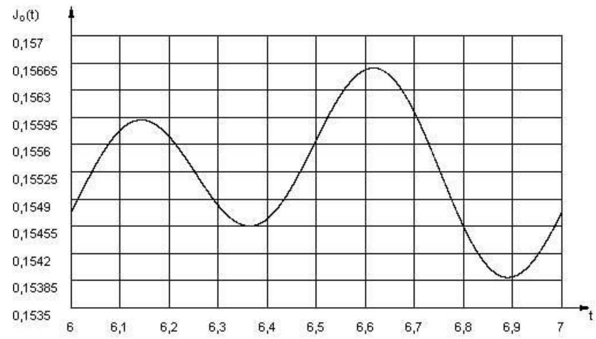


**Figure 8: Situation (iii), approximation of the limit behavior of** $J_0(t) = \Pr(X(t) = 0)$

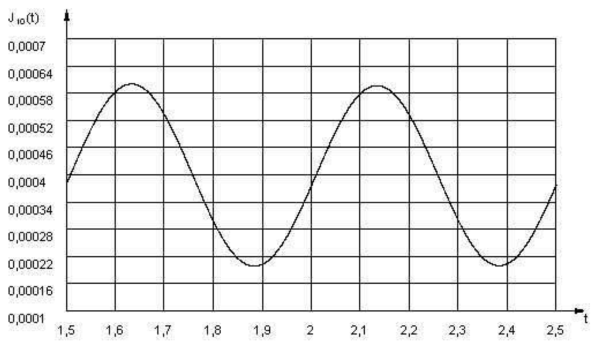


**Figure 6: Situation (ii), approximation of the limit behavior of** $J_{10}(t) = \Pr(X(t) \leq 10)$

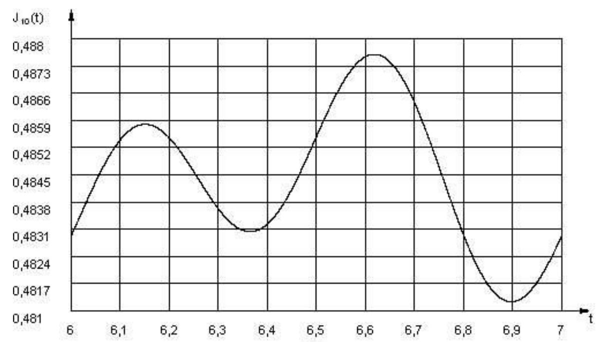


**Figure 9: Situation (iii), approximation of the limit behavior of** $J_{10}(t) = \Pr(X(t) \leq 10)$