# Stability bounds for $M_t/M_t/N/N+R$ queue


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## ABSTRACT

We study $M_t/M_t/N/N+R$ queue and obtain stability bounds for main characteristics of the respective queue-length process.



## Keywords

Nonstationary Markovian queueing model, stability, weak ergodicity, bounds


## 1. INTRODUCTION

Nonstationary Erlang loss queueing model has been studied in some recent papers, see [2,3,9]. Here we consider the simplest generalization of this model, namely we study nonstationary Markovian queue with $N$ servers and $R \geq 0$ waiting rooms and obtain the stability bounds for some characteristics of this queue. There is a number of investigations of stability for nonstationary continuous-time Markov chains, see for instance first results in [6], and more detail studies for birth and death processes (BDPs) in [1,7]. Here we apply our general approach and the idea of paper [5] and prove some simple stability bounds for nonstationary $M_t/M_t/N/N+R$ queue.

Let $X = X(t)$, $t \geq 0$ be queue-length process for $M_t/M_t/N/N+R$ queue. This is a BDP on state space $E_{N+R} = \{0, 1 \ldots, N+R\}$ and birth and death rates $\lambda_n(t) = \lambda(t)$, $\mu_n(t) = \min(n, N)\,\mu(t)$ respectively. We suppose that arrival and service intensities $\lambda(t)$ and $\mu(t)$ are locally integrable on $[0, \infty)$. Let $p_i(t) = Pr\{X(t) = i\}$ be state probabilities of $X(t)$, and $\mathbf{p}(t) = (p_0(t), \ldots, p_{N+R}(t))^T$ be the respective column vector.

Then we can write the forward Kolmogorov system

$$\begin{cases} \frac{dp_0}{dt} = -\lambda(t)p_0 + \mu(t)p_1, \\ \frac{dp_k}{dt} = \lambda(t)p_{k-1} - (\lambda(t) + k\mu(t))\,p_k + \\ \quad (k+1)\mu(t)p_{k+1}, 1 \leq k \leq N-1, \\ \frac{dp_k}{dt} = \lambda(t)p_{k-1} - (\lambda(t) + N\mu(t))\,p_k + \\ \quad N\mu(t)p_{k+1}, N \leq k < N+R, \\ \frac{dp_{N+R}}{dt} = \lambda(t)p_{N+R-1} - N\mu(t)p_{N+R} \end{cases} \quad (1)$$

in the following form:

$$\frac{d\mathbf{p}}{dt} = A\,(t)\,\mathbf{p}, \quad t \geq 0, \quad (2)$$

where $A(t) = \{a_{ij}(t),\ t \geq 0\}$ is the transposed intensity matrix of the process, and

$$a_{ij}(t) = \begin{cases} \lambda\,(t)\,, & \text{if} \quad j = i-1, \\ \min\,(i+1, N)\,\mu\,(t)\,, & \text{if} \quad j = i+1, \\ -\,(\lambda\,(t) + \min\,(i, N)\,\mu\,(t))\,, & \text{if} \quad j = i, \\ 0, & \text{overwise.} \end{cases} \quad (3)$$

We denote throughout the paper by $\|\bullet\|$ the $l_1$-norm, i.e. $\|\mathbf{x}\| = \sum |x_i|$, for $\mathbf{x} = (x_0, ..., x_{N+R})^T$ and $\|B\| = \max_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^{N+R}$.

Let $\Omega = \{\mathbf{x} : \mathbf{x} \geq 0, \|\mathbf{x}\| = 1\}$ be a set of all stochastic vectors.

Let $E_k(t) = E\{X(t)\,|X(0) = k\}$ be the mean of the process at the moment $t$ under initial condition $X(0) = k$, and $E_{\mathbf{p}}(t)$ be the mathematical expectation (the mean) at the moment $t$ under initial probability distribution $\mathbf{p}(0) = \mathbf{p}$.

Consider also a "perturbed" queue-length process $\bar{X} = \bar{X}(t)$, $t \geq 0$ with *general* structure of intensity matrix $\bar{A}(t)$. Namely, $\bar{X}(t)$ is not BDP in general. Put $\hat{A}(t) = \bar{A}(t) - A(t)$. We assume that the perturbations are uniformly small, i.e. $\|\hat{A}(t)\| \leq \varepsilon$ for almost all $t \geq 0$.

## 2. GENERAL STABILITY BOUNDS

Let $X(t)$ be a general BDP with finite state space $E_{N+R} = \{0, 1 \ldots, N+R\}$.

Let $d_1, ..., d_{N+R}$ be positive numbers. Put

$$\alpha_k\,(t) = \lambda_{k-1}\,(t) + \mu_k\,(t) - \frac{d_{k+1}}{d_k}\lambda_k\,(t) - \frac{d_{k-1}}{d_k}\mu_{k-1}\,(t)\,, \quad 1 \leq k \leq N+R, \quad (4)$$

where $d_0 = d_{N+R+1} = 0$.

Denote $G = \sum_{i=1}^{N+R} d_i$, $\quad d = \min_{1 \leq i \leq N+R} d_i$.

THEOREM 1. *Let there exist a positive sequence $\{d_i\}$ and a positive number $\theta$ such that*

$$\alpha_i(t) \geq \theta, \quad i = 1, 2, \dots, N+R, \; t \geq 0. \tag{5}$$

*Then the following stability bounds hold:*

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \leq \frac{\varepsilon\left(1 + \log\frac{4G}{d}\right)}{\theta}, \tag{6}$$

*and*

$$\limsup_{t\to\infty} \left|E_{\mathbf{p}}(t) - \bar{E}_{\bar{\mathbf{p}}}(t)\right| \leq \frac{(N+R)\,\varepsilon\left(1 + \log\frac{4G}{d}\right)}{\theta}, \tag{7}$$

*for arbitrary initial probability distributions $\mathbf{p}(0)$ and $\bar{\mathbf{p}}(0)$ for $X(t)$ and $\bar{X}(t)$ respectively.*

**Proof.** Firstly we obtain the bounds on the rate of convergence. The property $\sum_{i=0}^{N+R} p_i(t) = 1$ for any $t \geq 0$ allows to put $p_0(t) = 1 - \sum_{i\geq 1} p_i(t)$, then we obtain the following system from (2)

$$\frac{d\mathbf{z}(t)}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{8}$$

where $\mathbf{z}(t) = \left(p_1(t), \dots, p_{N+R}(t)\right)^T$, $\mathbf{f}(t) = (\lambda_0(t), 0, \dots, 0)^T$, and $B = (b_{ij})_{i,j=1}^{N+R} =$

$$\begin{pmatrix} -(\lambda_0+\lambda_1+\mu_1) & (\mu_2-\lambda_0) & -\lambda_0 & \cdots & -\lambda_0 \\ \lambda_1 & -(\lambda_2+\mu_2) & \mu_3 & \cdots & 0 \\ 0 & \lambda_2 & -(\lambda_3+\mu_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdot & \lambda_{N+R-1} & -\mu_{N+R} \end{pmatrix}. \tag{9}$$

Then we have

$$\mathbf{z}(t) \;=\; V(t,s)\mathbf{z}(s) + \int_s^t V(t,z)\mathbf{f}(z)\,dz, \tag{10}$$

where $V(t,z)$ is a Cauchy matrix for equation (8).

Consider now the triangular matrix

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots & d_1 \\ 0 & d_2 & d_2 & \cdots & d_2 \\ 0 & 0 & d_3 & \cdots & d_3 \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & d_{N+R} \end{pmatrix}, \tag{11}$$

and the respective norms $\|\mathbf{x}\|_{1D} = \|D\mathbf{x}\|$, and $\|B\|_{1D} = \|DBD^{-1}\|$.

We have $DB(t)D^{-1} =$

$$\begin{pmatrix} -(\lambda_0+\mu_1) & \frac{d_1}{d_2}\mu_1 & \cdots & 0 \\ \frac{d_2}{d_1}\lambda_1 & -(\lambda_1+\mu_2) & \cdots & 0 \\ 0 & \frac{d_3}{d_2}\lambda_2 & \ddots & 0 \\ \vdots & 0 & \ddots & 0 \\ & \cdots & \ddots & \frac{d_{N+R-1}}{d_{N+R}}\mu_{N+R-1} \\ 0 & \cdot & \frac{d_{N+R}}{d_{N+R-1}}\lambda_{N+R-1} & -(\lambda_{N+R-1}+\mu_{N+R}) \end{pmatrix} \tag{12}$$

and the following bound of the logarithmic norm $\gamma\left(B(t)\right)$ in $1D-$norm holds (see for instance [3,4,8,9]):

$$\gamma\left(B\right)_{1D} = \max_i\left(\frac{d_{i+1}}{d_i}\lambda_i(t) + \frac{d_{i-1}}{d_i}\mu_{i-1}(t) - \left(\lambda_{i-1}(t)+\mu_i(t)\right)\right) = \max\left(-\alpha_i\left(t\right)\right) \leq -\theta, \tag{13}$$

in accordance with (5). Therefore the following inequality holds:

$$\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\|_{1D} \leq e^{-\theta(t-s)}\|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D}, \tag{14}$$

for any initial conditions $\mathbf{z}^*(s)$, $\mathbf{z}^{**}(s)$ and any $s, t$, $0 \leq s \leq t$.

Then we obtain the following bound in 'natural' norm:

$$\begin{aligned} \|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 2\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\| = \\ 2\|D^{-1}D\left(\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\right)\| \leq \\ \frac{4}{d}\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\|_{1D} \leq \\ \frac{4}{d}e^{-\theta(t-s)}\|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\|_{1D} \leq \\ \frac{4G}{d}e^{-\theta(t-s)}\|\mathbf{z}^*(s) - \mathbf{z}^{**}(s)\| \leq \\ \frac{4G}{d}e^{-\theta(t-s)}\|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\| \leq \frac{8G}{d}e^{-\theta(t-s)}, \end{aligned} \tag{15}$$

for any initial conditions $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$ and any $s, t$, $0 \leq s \leq t$.

Consider now the forward Kolmogorov system for perturbed process:

$$\frac{d\bar{\mathbf{p}}}{dt} = \bar{\mathbf{A}}(t)\bar{\mathbf{p}}(t) \tag{16}$$

Here we slightly modify the approach of paper [5]. Put

$$\begin{aligned} \beta(t,s) = \sup_{\|\mathbf{v}\|=1, \sum v_i = 0}\|U(t)\mathbf{v}\| = \\ \tfrac{1}{2}\max_{i,j}\sum_k |p_{ik}(t,s) - p_{jk}(t,s)|, \end{aligned} \tag{17}$$

where $U(t,s)$ is Cauchy matrix of equation (2), and $p_{ik}(t,s) = Pr\left\{X(t) = k | X(s) = i\right\}$.

Then

$$\|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \leq \beta(t,s)\|\mathbf{p}(s) - \bar{\mathbf{p}}(s)\| + \int_s^t \|\hat{A}(u)\|\beta(u,s)du. \tag{18}$$

Moreover, the following estimates hold:

$$\beta(t,s) \leq 1, \quad \beta(t,s) \leq \frac{ce^{-b(t-s)}}{2}, \; 0 \leq s \leq t, \tag{19}$$

where $c = \frac{8G}{d}$, $b = \theta$.

Finally we have

$$\|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \leq \left\{\begin{array}{c} \|\mathbf{p}(s) - \bar{\mathbf{p}}(s)\| + (t-s)\varepsilon, \quad 0 < t < b^{-1}\log\frac{c}{2}, \\ b^{-1}(\log\frac{c}{2} + 1 - ce^{-b(t-s)})\varepsilon + \frac{c}{2}e^{-b(t-s)}\|\mathbf{p}(s) - \bar{\mathbf{p}}(s)\|, \\ t \geq b^{-1}\log\frac{c}{2} \end{array}\right. \tag{20}$$

for any initial conditions $\mathbf{p}(s)$ and $\bar{\mathbf{p}}(s)$. Hence for $s = 0$ and $t \to \infty$ we obtain (6).

The second bound (7) follows from the inequality

$$\left|E_{\mathbf{p}}(t) - \bar{E}_{\bar{\mathbf{p}}}(t)\right| \leq \sum_k k|p_k(t) - \bar{p}_k(t)| \leq (N+R)\|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|.$$

COROLLARY 1. *Let $\lambda(t)$ and $\mu(t)$ be 1-periodic. Let (instead of (5)) there exist a positive sequence $\{d_i\}$ and a positive number $\varphi^*$ such that*

$$\alpha_i(t) \geq \varphi(t), \quad i = 1, 2, \dots, N+R, 0 \leq t \leq 1, \tag{21}$$

*where*

$$\int_0^1 \varphi(t)\,dt = \varphi^*. \tag{22}$$

*Let*

$$K = \sup_{|t-s|\le 1} \int_s^t \varphi(\tau)\,d\tau < \infty. \tag{23}$$

*Then we have the following stability bounds:*

$$\limsup_{t\to\infty} \|\mathbf{p}(t)-\bar{\mathbf{p}}(t)\| \le \frac{\varepsilon\left(1+\log\frac{4Ge^K}{d}\right)}{\varphi^*}, \tag{24}$$

*and*

$$\limsup_{t\to\infty} \left|E_{\mathbf{p}}(t)-\bar{E}_{\bar{\mathbf{p}}}(t)\right| \le \frac{(N+R)\,\varepsilon\left(1+\log\frac{4Ge^K}{d}\right)}{\varphi^*}, \tag{25}$$

*for arbitrary initial probability distributions* $\mathbf{p}(0)$ *and* $\bar{\mathbf{p}}(0)$ *for* $X(t)$ *and* $\bar{X}(t)$ *respectively.*

**Proof.** The statement follows from inequality $e^{-\int_s^t \varphi(u)\,du} \le e^K e^{-\varphi^*(t-s)}$.

We can use another approach to bounding the rate of convergence in 'natural' norm, namely, in the final part of (16) we have

$$\|\mathbf{z}^*(s)-\mathbf{z}^{**}(s)\|_{1D} \le \|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|_{1D}. \tag{26}$$

Put $s=0$, $\mathbf{p}^*(0)=\pi(0)$, $\mathbf{p}^{**}(0)=\mathbf{p}(0)=e_0$, where $\pi(t)$ is 1-periodic. Then we obtain $\|\pi(0)\|_{1D} \le \limsup_{t\to\infty}\|\pi(t)\|_{1D}$ and

$$\|\pi(t)\|_{1D} \le \|V(t,0)\pi(0)\|_{1D} + \left\|\int_0^t V(t,\tau)\mathbf{f}(\tau)d\tau\right\|_{1D} \le$$
$$\le e^K e^{-\varphi^* t}\|\pi(0)\|_{1D} + M_1\int_0^t e^{-\int_\tau^t \varphi(u)\,du}d\tau \tag{27}$$
$$\le e^K e^{-\varphi^* t}\|\pi(0)\|_{1D} + M_1 e^K\int_0^t e^{-\varphi^*(t-\tau)}d\tau,$$

where $e^{-\int_0^t \varphi(u)du} \le e^K e^{-\varphi^* t}$ and $\lambda_0(t) \le M_1$ for almost all $t \ge 0$. Then

$$\limsup_{t\to\infty}\|\pi(t)\|_{1D} \le \frac{M_1 e^K}{\varphi^*}. \tag{28}$$

Therefore in (19) and (20) we have $c = \frac{4e^{2K}M_1}{d\varphi^*}$, $\quad b=\varphi^*$ and choosing $\bar{p}(0)=\bar{\pi}(0)$, we obtain the following statement.

Corollary 2. *Let* $\lambda_0(t) \le M_1$ *for almost all* $t \ge 0$*, and let the assumptions of Corollary 1 be fulfilled. Then the following bounds hold:*

$$\limsup_{t\to\infty} \|\mathbf{p}(t)-\bar{\mathbf{p}}(t)\| \le \frac{\varepsilon\left(1+\log\frac{2e^{2K}M_1}{d\varphi^*}\right)}{\varphi^*}, \tag{29}$$

*and*

$$\limsup_{t\to\infty} \left|E_{\mathbf{p}}(t)-\bar{E}_{\bar{\mathbf{p}}}(t)\right| \le \frac{\varepsilon(N+R)\left(1+\log\frac{2e^{2K}M_1}{d\varphi^*}\right)}{\varphi^*}. \tag{30}$$

Now we consider essentially another approach.
Denote

$$W = \min_{i\ge 1}\frac{d_i}{i}, \quad m = \max_{|i-j|=1}\frac{d_i}{d_j}. \tag{31}$$

Theorem 2. *Let the assumptions of Corollary 2 be fulfilled. Then the following stability bound holds:*

$$\limsup_{t\to\infty}\left|E_{\mathbf{p}}(t)-\bar{E}_{\mathbf{p}}(t)\right| \le \tag{32}$$
$$\frac{2e^K\varepsilon e^{(1+m)\varepsilon}}{W\varphi^*}\left((1+m)\frac{M_1e^K}{\varphi^*}+\frac{d_1}{2}\right).$$

**Proof.** Rewrite system (8) in the following form:

$$\frac{d\mathbf{z}}{dt} = \bar{B}(t)\mathbf{z}(t)+\bar{f}(t)+\hat{B}(t)\mathbf{z}(t)+\hat{f}(t), \tag{33}$$

where $\hat{B}(t)=B(t)-\bar{B}(t)$, $\hat{f}(t)=f(t)-\bar{f}(t)$.
Then in *any* norm the following bound holds:

$$\|\mathbf{z}(t)-\bar{\mathbf{z}}(t)\| \le \int_0^t \|\bar{V}(t,\tau)\|(\|\hat{B}(\tau)\|\|\mathbf{z}(\tau)\|+\|\hat{f}(t)\|)\,d\tau, \tag{34}$$

if the initial conditions $\mathbf{z}(0)=\bar{\mathbf{z}}(0)$ are the same.
We have

$$\|\hat{B}(t)\|_{1D} = \|D\hat{B}(t)D^{-1}\|_1 \le$$
$$\max_n\left(\tfrac{\varepsilon}{2}(1+\tfrac{d_{n+1}}{d_n})+\tfrac{\varepsilon}{2}(1+\tfrac{d_{n-1}}{d_n})\right) \le (1+m)\,\varepsilon. \tag{35}$$

Therefore

$$\gamma(\bar{B}(t))_{1D} \le \gamma(DB(t)D^{-1})_1 + \|\hat{B}(t)\|_{1D} \le -\varphi(t)+(1+m)\,\varepsilon. \tag{36}$$

On the other hand 1-periodicity of $\mathbf{z}(t)$ and $\pi(t)$ implies the inequality $\|\mathbf{z}(t)\|_{1D} \le \|\pi(t)\|_{1D} \le \limsup_{t\to\infty}\|\pi(t)\|_{1D}$, and we can apply bound (28).
Moreover,

$$\|\mathbf{z}\|_{1E} = \textstyle\sum_{k\ge1} k|p_k| = \sum_{k\ge1}\frac{k}{d_k}d_k|p_k| \le$$
$$W^{-1}\textstyle\sum_{k\ge1} d_k|p_k| = W^{-1}\sum_{k\ge1} d_k\left|\sum_{i\ge k}p_i - \sum_{i\ge k+1}p_i\right| \le$$
$$W^{-1}\textstyle\sum_{k\ge1} d_k\left(\left|\sum_{i\ge k}p_i\right| + \left|\sum_{i\ge k+1}p_i\right|\right) \le$$
$$\tfrac{2}{W}\textstyle\sum_{k\ge1} d_k\left|\sum_{i\ge k}p_i\right| \le \tfrac{2}{W}\|\mathbf{z}\|_{1D}. \tag{37}$$

Note that $\|\hat{f}(t)\|_{1D} = \frac{d_1\varepsilon}{2}$.
Hence we have

$$\left|E_{\mathbf{p}}(t)-\bar{E}_{\mathbf{p}}(t)\right| \le \|\mathbf{z}(t)-\bar{\mathbf{z}}(t)\|_{1E} \le \tfrac{2}{W}\|\mathbf{z}(t)-\bar{\mathbf{z}}(t)\|_{1D} \le$$
$$\le \tfrac{2\varepsilon}{W}\left((1+m)\tfrac{M_1e^K}{\varphi^*}+\tfrac{d_1}{2}\right)\int_0^t e^{-\int_\tau^t(\varphi(u)-(1+m)\varepsilon)\,du}d\tau \le$$
$$\le \tfrac{2e^K\varepsilon e^{(1+m)\varepsilon}}{W\varphi^*}\left((1+m)\tfrac{M_1e^K}{\varphi^*}+\tfrac{d_1}{2}\right). \tag{38}$$

## 3. BOUNDS FOR THE QUEUE-LENGTH PROCESS

Let now $X(t)$ be a queue-length process for $M_t/M_t/N/N+R$ queue. Then we have

$$\alpha_k(t) = \lambda(t)+k\mu(t)-\frac{d_{k+1}}{d_k}\lambda(t)-\frac{d_{k-1}}{d_k}(k-1)\mu(t),$$

if $1 \le k \le N$, and

$$\alpha_k(t) = \lambda(t) + N\mu(t) - \frac{d_{k+1}}{d_k}\lambda(t) - \frac{d_{k-1}}{d_k}N\mu(t),$$

if $N < k \le N+R$.

**First case, large service rate.**

Let firstly there exist $l > 1$ such that

$$N\mu(t) - l\lambda(t) \ge \omega > 0, \tag{39}$$

for almost all $t \ge 0$. Put $d_1 = 1$, $\frac{d_{k+1}}{d_k} = 1,\ k \le N-2$, and $\frac{d_{k+1}}{d_k} = l,\ k \ge N-1$.

Then

$$\alpha_k(t) = \begin{cases} \mu(t), & k < N-1; \\ \mu(t) - (l-1)\lambda(t), & k = N-1; \\ \left(1-\frac{1}{l}\right)(N\mu(t) - l\lambda(t)), & N \le k \le N+R-1; \\ N\mu(t)\left(1-\frac{1}{l}\right) + \lambda(t), & k = N+R. \end{cases} \tag{40}$$

Suppose $l \le \frac{N}{N-1}$, hence

$$\varphi(t) = \min_k \alpha_k(t) = \left(1 - \frac{1}{l}\right)(N\mu(t) - l\lambda(t)). \tag{41}$$

Proposition 1. *Let (39) be satisfied. Then stability estimates (6) and (7) hold, where* $\theta = (1-\frac{1}{l})\omega$, $d = 1$ *and* $G = N - 1 + \sum_{i=1}^{R+1} l^i$.

Proposition 2. *Let arrival and service rates* $\lambda(t)$ *and* $\mu(t)$ *be 1-periodic. Let (instead of (39)) there exist* $\zeta$ *such that*

$$\int_0^1 (N\mu(t) - l\lambda(t))\,dt \ge \zeta > 0. \tag{42}$$

*Then bounds (24) and (25) hold, where* $\varphi^* = (1-\frac{1}{l})\zeta$, $d = 1$ *and* $G = N - 1 + \sum_{i=1}^{R+1} l^i$.

Suppose now $l \ge \frac{N}{N-1}$.

Proposition 3. *Let arrival and service rates* $\lambda(t)$ *and* $\mu(t)$ *be 1-periodic,* $\lambda(t) \le M_1$ *for almost all* $t \in [0,1]$. *Let there exist* $l > 1$ *such that*

$$\min_k \alpha_k = \mu(t), \int_0^1 \mu(t)\,dt \ge \psi > 0, K = \sup_{|t-s|\le 1} \int_s^t \mu(\tau)d\tau. \tag{43}$$

*Then the following bounds hold:*

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \le \frac{\varepsilon\left(1 + \log\frac{2e^{2K}M_1}{\psi}\right)}{\psi}, \tag{44}$$

$$\limsup_{t\to\infty} \left|E_{\mathbf{p}}(t) - \bar{E}_{\mathbf{p}}(t)\right| \le \frac{2\varepsilon(N-1)e^K e^{(1+l)\varepsilon}}{\psi}\left((1+l)\frac{M_1 e^K}{\psi} + \frac{1}{2}\right). \tag{45}$$

**Proof.** Bound (44) follows from Corollary 2 for $d = 1$ and $\varphi^* = \psi$. Bound (45) follows from Theorem 2 for $d = 1$, $\varphi^* = \psi$, $m = l$ and $W = \frac{1}{N-1}$.

**Second case, large arrival rate.**

Let firstly for some $l < 1$ the following inequality holds:

$$l\lambda(t) - N\mu(t) \ge \omega > 0 \tag{46}$$

Put $\frac{d_{k+1}}{d_k} = l,\ k \ge 1$. Then

$$\alpha_k(t) = \begin{cases} \left(\frac{1}{l}-1\right)(l\lambda(t) - k\mu(t)) + \mu(t), & k \le N-1; \\ \left(\frac{1}{l}-1\right)(l\lambda(t) - N\mu(t)), & N \le k \le N+R-1; \\ \lambda(t) - N\left(\frac{1}{l}-1\right)\mu(t), & k = N+R \end{cases} \tag{47}$$

and

$$\varphi(t) = \min_k \alpha_k(t) = \left(\frac{1}{l} - 1\right)(l\lambda(t) - N\mu(t)). \tag{48}$$

Proposition 4. *Let* (46) *be fulfilled. Then stability estimates (6) and (7) hold, where* $\theta = (\frac{1}{l}-1)\omega$, $d = l^{N+R}$ *and* $G < N+R$.

Proposition 5. *Let now* $\lambda(t)$ *and* $\mu(t)$ *be 1-periodic. Let for some positive* $\zeta$

$$\int_0^1 (l\lambda(t) - N\mu(t))\,dt \ge \zeta > 0. \tag{49}$$

*Then the following stability bounds hold:*

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \le \frac{\varepsilon\left(1 + \log\frac{4e^K(N+R)}{l^{N+R}}\right)}{\left(\frac{1}{l}-1\right)\zeta}, \tag{50}$$

*and*

$$\limsup_{t\to\infty} |E_{\mathbf{p}}(t) - \bar{E}_{\bar{\mathbf{p}}}(t)| \le \frac{(N+R)\varepsilon\left(1 + \log\frac{4e^K(N+R)}{l^{N+R}}\right)}{\left(\frac{1}{l}-1\right)\zeta}. \tag{51}$$

# 4. EXAMPLES

Example 1. *Let* $\lambda(t) = 9 + \sin 2\pi t$, $\mu(t) = 1 + \cos 2\pi t$, $N = 100$, $R = 10^5$, $\varepsilon = 10^{-6}$.

*The assumptions of Proposition 3 are fulfilled for* $l = 2$. *Then* $M_1 = 10$, $K = 1 + \frac{1}{\pi}$, $\psi = 1$ *and we have the following stability bounds:*

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \le 6.632 \cdot 10^{-6} \tag{52}$$

*and*

$$\limsup_{t\to\infty} |E_{\mathbf{p}}(t) - \bar{E}_{\bar{\mathbf{p}}}(t)| \le 0.084. \tag{53}$$

*Hence we can apply the approach of [8] and find the limit characteristics approximately with the same error* $\varepsilon$ *as the respective characteristics of truncated process with* $m = 146$ *and* $t \in [21.0, 22.0]$. *The corresponding graphs are shown in Figures 1-2.*

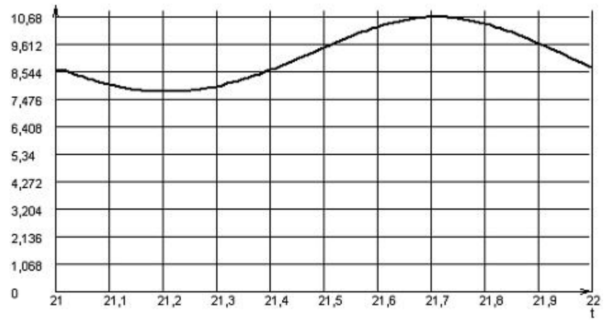


**Figure 1: Approximation of the limiting mean $\bar{E}_{\bar{\mathbf{p}}}(t)$.**

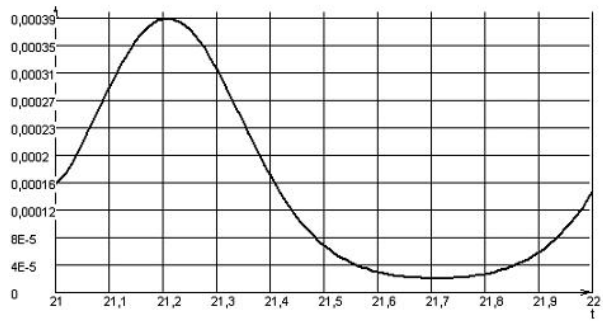


**Figure 2: Approximation of the limit behavior of $\bar{J}_0(t) = \Pr\left(\bar{X}(t) = 0\right)$.**

EXAMPLE 2. *Let* $\lambda(t) = 250 + 200\sin 2\pi t$, $\mu(t) = 1 + \cos 2\pi t$, $N = 100$, $R = 10^4$, $\varepsilon = 10^{-6}$.

*Then the assumptions of Proposition 5 are satisfied for* $l = \frac{1}{2}$. *We have* $\int_0^1 (l\lambda(t) - N\mu(t))\,dt = 25$, $M_1 = 450$, $K = 100 + \frac{101}{\pi}$, $\psi = 1$. *Hence the following stability bounds hold:*

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \le 2.807 \cdot 10^{-4}, \qquad (54)$$

$$\limsup_{t\to\infty} |E_{\mathbf{p}}(t) - \bar{E}_{\bar{\mathbf{p}}}(t)| \le 2.836. \qquad (55)$$

## 5. ACKNOWLEDGMENTS

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