On stability for $M_t/M_t/N/N$ queue

Alexander Zeifman
Vologda State Pedagogical University
Institute of Informatics Problems RAS
and ISEDT RAS Email: a_zeifman@mail.ru

Anna Korotysheva
Vologda State Pedagogical University
Email: a_korotysheva@mail.ru

Yakov Satin
Vologda State Pedagogical University
Email: yakovi@mail.ru

***Abstract*—We obtain some stability bounds for nonstationary Erlang loss queueing model. Two specific examples of queues with close to periodic arrival and service rates are considered.**

***Index Terms*—queueing system; stability bounds; nonstationary Erlang model**

I. INTRODUCTION

In this note we consider the estimates of stability for the simplest nonstationary Erlang queueing model. There is a number of investigations of nonstationary continuous-time Markov chains, see for instance first results in [5], and more detail studies for birth and death processes (BDPs) in [1], [6]. Now we consider nonstationary $M_t/M_t/N/N$ queue and obtain some new and simple stability bounds.

Let $X = X(t),\ t \geq 0$ be queue-length process for $M_t/M_t/N/N$ queue. This is a BDP on state space $E_N = \{0, 1 \ldots, N\}$ and birth and death rates $\lambda_n(t) = \lambda(t)$, $\mu_n(t) = n\mu(t)$ respectively. We suppose that arrival and service intensities $\lambda(t)$ and $\mu(t)$ are locally integrable on $[0, \infty)$. Let $p_i(t) = Pr\{X(t) = i\}$ be state probabilities of $X(t)$, and $\mathbf{p(t)} = (\mathbf{p_0(t)}, \ldots, \mathbf{p_N(t)})^{\mathbf{T}}$ be the respective column vector.

Then we have the forward Kolmogorov system

$$\begin{cases} \frac{dp_0}{dt} = -\lambda(t)p_0 + \mu(t)p_1, \\ \frac{dp_k}{dt} = \lambda(t)p_{k-1} - (\lambda(t) + k\mu(t))\, p_k + (k+1)\mu(t)p_{k+1}, \\ 1 \leq k < N, \\ \frac{dp_N}{dt} = \lambda(t)p_{N-1} - N\mu(t)p_N \end{cases} \tag{1}$$

in the following form:

$$\frac{d\mathbf{p}}{dt} = \mathbf{A}\,(\mathbf{t})\,\mathbf{p}, \quad \mathbf{t} \geq \mathbf{0}, \tag{2}$$

where $\mathbf{A(t)} = \{\mathbf{a_{ij}(t)},\ \mathbf{t} \geq \mathbf{0}\}$ is the transposed intensity matrix of the process, and

$$a_{ij}(t) = \begin{cases} \lambda\,(t)\,, & \text{if} \quad j = i-1, \\ (i+1)\mu\,(t)\,, & \text{if} \quad j = i+1, \\ -\left(\lambda\,(t) + i\mu\,(t)\right), & j = i, \\ 0, & \text{overwise}. \end{cases} \tag{3}$$

We denote throughout the paper by $\|\bullet\|$ the l_1-norm, i.e. $\|\mathbf{x}\| = \sum |\mathbf{x_i}|$, for $\mathbf{x} = (\mathbf{x_0}, ..., \mathbf{x_N})^{\mathbf{T}}$ and $\|B\| = \max_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^N$.

Let $\Omega = \{\mathbf{x} : \mathbf{x} \geq 0,\ \|\mathbf{x}\| = 1\}$ be a set of all stochastic vectors.

Let $E_k(t) = E\{X(t)\,|X(0) = k\}$ be the mean of the process at the moment t under initial condition $X(0) = k$, and $E_{\mathbf{p}}(t)$ be the mathematical expectation (the mean) at the moment t under initial probability distribution $\mathbf{p}(0) = \mathbf{p}$.

Consider also a "perturbed" queue-length process $\bar{X} = \bar{X}(t),\ t \geq 0$ with general structure of intensity matrix $\bar{A}(t)$. In general, $\bar{X}(t)$ is not BDP. Put $\hat{A}(t) = \bar{A}(t) - A(t)$. We assume that the perturbations are uniformly small, i.e. $\|\hat{A}(t)\| \leq \varepsilon$ for almost all $t \geq 0$.

II. STABILITY BOUNDS

Let $d_1, \ldots, d_N$ be positive numbers. Consider the following expression:

$$\alpha_i(t) = \lambda(t) + i\mu(t) - \frac{d_{i+1}}{d_i}\lambda(t) - \frac{d_{i-1}}{d_i}(i-1)\mu(t), \quad i = 1, 2, \ldots, N, \tag{4}$$

where $d_0 = d_{N+1} = 0$. Put $G = \sum_{i=1}^N d_i$ and $d = \min_i d_i$.

Theorem 1. *Let there exist a positive sequence $\{d_i\}$ and a positive number β such that*

$$\alpha_i(t) \geq \beta, \quad i = 1, 2, \ldots, N,\ t \geq 0. \tag{5}$$

Then the following stability bounds hold:

$$\limsup_{t\to\infty} \|\mathbf{p(t)} - \bar{\mathbf{p}}(\mathbf{t})\| \leq \frac{\varepsilon\left(1 + \log\frac{4G}{d}\right)}{\beta}, \tag{6}$$

and

$$\limsup_{t\to\infty} \left|E_{\mathbf{p}}(t) - \bar{E}_{\bar{\mathbf{p}}}(t)\right| \leq \frac{N\varepsilon\left(1 + \log\frac{4G}{d}\right)}{\beta}, \tag{7}$$

for arbitrary initial probability distributions $\mathbf{p(0)}$ and $\bar{\mathbf{p}}(0)$ for $X(t)$ and $\bar{X}(t)$ respectively.

Proof. Firstly we find the basic estimate of the rate of convergence. The property $\sum_{i=0}^N p_i(t) = 1$ for any $t \geq s$ allows to put $p_0(t) = 1 - \sum_{i\geq 1} p_i(t)$, then we obtain the following system from (2)

$$\frac{d\mathbf{z}(t)}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{8}$$

978-1-4244-7286-4/10/$26.00 ©2010 IEEE

where $\mathbf{z}(t)=(p_1(t),\dots,p_N(t))^T$, $\mathbf{f}(t)=(\lambda_0(t),0,\dots,0)^T$, $B(t)=(b_{ij}(t))_{i,j=1}^N$ and respective $b_{ij}(t)$, see details in [7], [8]. Consider now the triangular matrix

$$D=\begin{pmatrix} d_1 & d_1 & d_1 & \cdots & d_1\\ 0 & d_2 & d_2 & \cdots & d_2\\ 0 & 0 & d_3 & \cdots & d_3\\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & d_N \end{pmatrix}, \qquad (9)$$

and the respective norms $\|\mathbf{x}\|_{\mathbf{1D}}=\|\mathbf{Dx}\|$, and $\|B\|_{1D}=\|DBD^{-1}\|$.

We have now the following bound of the logarithmic norm $\gamma(B(t))$ in $1D-$norm (see for instance [2], [3], [7], [9]):

$$\gamma(B)_{1D}=\max_{i\geq 0}\left(\frac{d_{i+1}}{d_i}\lambda(t)+\frac{d_{i-1}}{d_i}i\mu(t)-\right.$$
$$\left.(\lambda(t)+(i+1)\mu(t))\right)=\max\left(-\alpha_i(t)\right)\leq-\beta, \qquad (10)$$

in accordance with (5). Therefore the following inequality holds:

$$\|\mathbf{z}^*(t)-\mathbf{z}^{**}(t)\|_{1D}\leq e^{-\beta(t-s)}\|\mathbf{z}^*(s)-\mathbf{z}^{**}(s)\|_{1D}, \qquad (11)$$

for any initial conditions $\mathbf{z}^*(s)$, $\mathbf{z}^{**}(s)$ and any $s,t,\ 0\leq s\leq t$. Then we obtain

$$\begin{aligned}\|\mathbf{p}^*(t)-\mathbf{p}^{**}(t)\|\leq 2\|\mathbf{z}^*(t)-\mathbf{z}^{**}(t)\|=\\ 2\|D^{-1}D\left(\mathbf{z}^*(t)-\mathbf{z}^{**}(t)\right)\|\leq\\ \frac{4}{d}\|\mathbf{z}^*(t)-\mathbf{z}^{**}(t)\|_{1D}\leq\\ \frac{4}{d}e^{-\beta(t-s)}\|\mathbf{z}^*(s)-\mathbf{z}^{**}(s)\|_{1D}\leq\\ \frac{4G}{d}e^{-\beta(t-s)}\|\mathbf{z}^*(s)-\mathbf{z}^{**}(s)\|\leq\\ \frac{4G}{d}e^{-\beta(t-s)}\|\mathbf{p}^*(s)-\mathbf{p}^{**}(s)\|\leq\frac{8G}{d}e^{-\beta(t-s)},\end{aligned} \qquad (12)$$

for any initial conditions $\mathbf{p}^*(s)$, $\mathbf{p}^{**}(s)$ and any $s,t,\ 0\leq s\leq t$.

Consider the forward Kolmogorov system for perturbed process:

$$\frac{d\bar{\mathbf{p}}}{dt}=\bar{A}(t)\bar{\mathbf{p}}(t). \qquad (13)$$

We can apply the approach of paper [4]. Put

$$\beta(t,s)=\sup_{\|\mathbf{v}\|=1,\sum v_i=0}\|U(t,s)\mathbf{v}\|=$$
$$\frac{1}{2}\max_{i,j}\sum_k|p_{ik}(t,s)-p_{jk}(t,s)|, \qquad (14)$$

where $U(t,s)$ is Cauchy matrix of (2), and $p_{ik}(t,s)=Pr\left\{X(t)=k|X(s)=i\right\}$. Mitrophanov in [4] proved the bound of stability, that in the nonstationary case is the following one:

$$\|\mathbf{p}(t)-\bar{\mathbf{p}}(t)\|\leq\beta(t,s)\|\mathbf{p}(s)-\bar{\mathbf{p}}(s)\|+\int_s^t\|\hat{A}(u)\|\beta(u,s)\,du. \qquad (15)$$

Moreover, the following estimates hold:

$$\beta(t,s)\leq 1,\quad \beta(t,s)\leq\frac{ce^{-b(t-s)}}{2},\ 0\leq s\leq t, \qquad (16)$$

where under our assumptions $c=\frac{8G}{d}$, $b=\beta$. Finally the following stability bound holds:

$$\|\mathbf{p}(t)-\bar{\mathbf{p}}(t)\|\leq\begin{cases}\|\mathbf{p}(s)-\bar{\mathbf{p}}(s)\|+(t-s)\varepsilon,\\ 0<t-s<b^{-1}\log\frac{c}{2},\\ b^{-1}(\log\frac{c}{2}+1-ce^{-b(t-s)})\varepsilon+\\ \frac{c}{2}e^{-b(t-s)}\|\mathbf{p}(s)-\bar{\mathbf{p}}(s)\|,\quad t-s\geq b^{-1}\log\frac{c}{2},\end{cases} \qquad (17)$$

for any initial conditions $\mathbf{p}(s)$, $\bar{\mathbf{p}}(s)$. Let $t-s\to\infty$. Then (17) implies our claim.

Let now non-perturbed process has 1-periodic intensities. Then we can obtain the following stability result.

Theorem 2. *Let $\lambda(t)$ and $\mu(t)$ be 1-periodic. Let there exist a positive sequence $\{d_i\}$ and a positive number β^* such that*

$$\alpha_i(t)\geq\beta(t),\quad i=1,2,\dots,N,0\leq t\leq 1, \qquad (18)$$

where

$$\int_0^1\beta(t)\,dt\geq\beta^*. \qquad (19)$$

Let

$$K=\sup_{|t-s|\leq 1}\int_s^t\beta(\tau)\,d\tau<\infty. \qquad (20)$$

Then the following stability bounds hold:

$$\limsup_{t\to\infty}\|\mathbf{p}(\mathbf{t})-\bar{\mathbf{p}}(\mathbf{t})\|\leq\frac{\varepsilon\left(1+\log\frac{4Ge^K}{d}\right)}{\beta^*}, \qquad (21)$$

and

$$\limsup_{t\to\infty}\left|E_{\mathbf{p}}(t)-\bar{E}_{\bar{\mathbf{p}}}(t)\right|\leq\frac{N\varepsilon\left(1+\log\frac{4Ge^K}{d}\right)}{\beta^*}, \qquad (22)$$

for arbitrary initial probability distributions $\mathbf{p}(\mathbf{0})$ and $\bar{\mathbf{p}}(0)$ for $X(t)$ and $\bar{X}(t)$ respectively.

Proof. We have

$$e^{-\int_s^t\beta(u)\,du}\leq Re^{-\beta^*(t-s)}, \qquad (23)$$

where $R=e^K$, and now $c=\frac{8Ge^K}{d}$, $b=\beta^*$.

III. Examples

Example 1.

Let $\lambda(t) = 2 + \sin 2\pi t$, $\mu(t) = 1 + \cos 2\pi t$, $\varepsilon = 10^{-6}$. Put all $d_i = 1$. Then we have

$$\beta(t) = \mu(t) = 1 + \cos 2\pi t, \quad G = N, \quad d = 1. \tag{24}$$

Therefore, instead of (6) and (7) we can write the following estimates

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 8Ne^{-\int_0^t (1+\cos 2\pi u)\, du} \leq 16Ne^{-t}, \tag{25}$$

and

$$\|E_{\mathbf{p}^*} - E_{\mathbf{p}^{**}}\| \leq 16N^2 e^{-t}. \tag{26}$$

Let $N = 5000$. We have the following stability bounds:

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \leq 1.3 \cdot 10^{-5}$$

and

$$\limsup_{t\to\infty} |E_{\mathbf{p}(t)} - E_{\bar{\mathbf{p}}(t)}| \leq 0.065.$$

Now, $16Ne^{-t} \leq 2.035 \cdot 10^{-8}$ and $16N^2e^{-t} \leq 1.02 \cdot 10^{-4}$ for $t \geq 29$, and we can find the limiting characteristics of original and perturbed processes approximately at interval $t \in [29, 30]$. Namely, we have limiting loss probability $\bar{p}_N(t) \approx p_N(t) \approx 0$, double mean $\bar{E} \approx E = lim_{t\to\infty} \frac{\int_0^t \phi(u)\, du}{t} \approx 2.103$, limiting probability of empty queue $\bar{p}_0(t) \approx p_0(t)$ is shown in Fig.1, and the limiting mean $\bar{\phi}(t)$ is shown in Fig.2.

Example 2.

Let now $\lambda(t) = N(2 + \sin 2\pi t)$, $\mu(t) = 1 + \cos 2\pi t$, $\varepsilon = 10^{-6}$. Put $d_i = 1$. Then we have

$$\beta(t) = 1 + \cos 2\pi t, \quad G = N, \quad d = 1. \tag{27}$$

Therefore, instead of (6) and (7) we can write the same bounds

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 8Ne^{-\int_0^t (1+\cos 2\pi u)\, du} \leq 16Ne^{-t}, \tag{28}$$

and

$$\|E_{\mathbf{p}^*} - E_{\mathbf{p}^{**}}\| \leq 16N^2 e^{-t}. \tag{29}$$

Let $N = 1000$. We have the following stability bounds:

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \leq 1.14 \cdot 10^{-5},$$

and

$$\limsup_{t\to\infty} |E_{\mathbf{p}(t)} - E_{\bar{\mathbf{p}}(t)}| \leq 0.012.$$

Now, $16Ne^{-t} \leq 4.07 \cdot 10^{-9}$, $16N^2e^{-t} \leq 4.0699 \cdot 10^{-6}$ for $t \geq 29$, and we can find the limiting characteristics of original and perturbed processes approximately at interval $t \in [29, 30]$.

Namely, we have limiting probability of empty queue $\bar{p}_0(t) \approx p_0(t) \approx 0$, double mean $\bar{E} \approx E = lim_{t\to\infty} \frac{\int_0^t \phi(u)\, du}{t} \approx 984.64$, limiting loss probability $\bar{p}_N(t) \approx p_N(t)$ is shown in Fig.3, and the limiting mean $\bar{\phi}(t) \approx \phi(t)$ is shown in Fig.4.

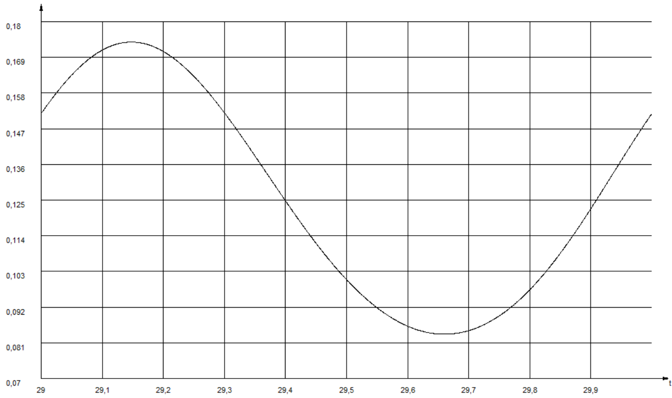

Fig. 1: Example 1. Approximation of the limit behavior of $\bar{p}_0(t) = \Pr\left(\bar{X}(t) = 0\right)$.

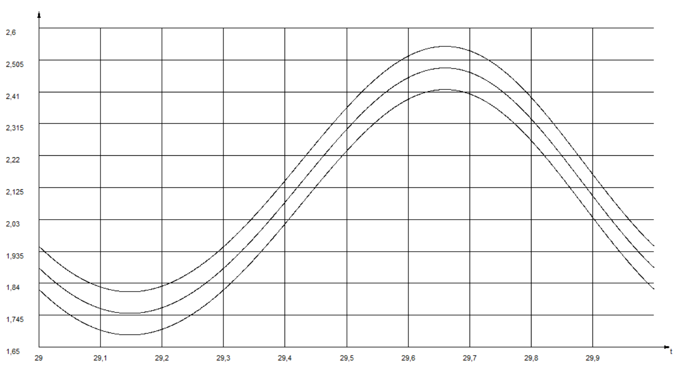

Fig. 2: Example 1. Approximation of the limiting mean $\bar{\phi}(t) = \bar{E}_0(t)$.

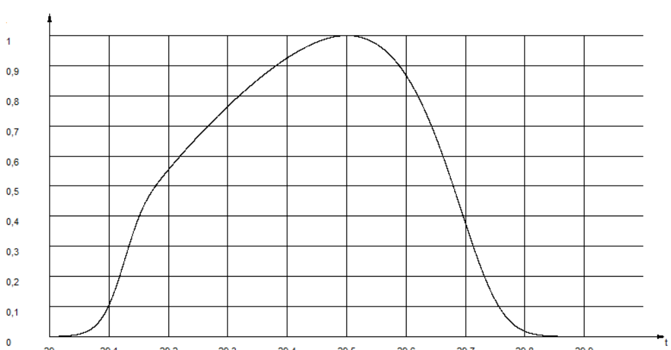

Fig. 3: Example 2. Approximation of the limit behavior of loss probability $\bar{p}_N(t)$.

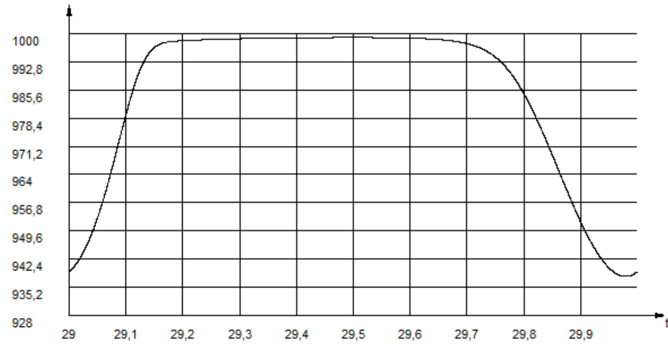

Fig. 4: Example 2. Approximation of the limiting mean $\bar{\phi}(t) = \bar{E}_0(t)$.

References

[1] Andreev D.B. at al. Ergodicity and stability of nonstationary queueing systems. *Theor. Probability and Math. Statist.*, 2004, **68**, 1–10.

[2] E. A. van Doorn, A. I. Zeifman, and T. L. Panfilova. Bounds and Asymptotics for the Rate of Convergence of Birth-Death Processes. *Theory of Probability and Its Applications*, 2010, **54**, 97–113.

[3] Granovsky B., Zeifman A. Nonstationary queues: Estimation of the rate of convergence. *Queueing Syst.*, 2004, **46**, 363–388.

[4] Mitrophanov A.Yu. Stability and exponential convergence of continuous-time Markov chains. *J. Appl. Prob.*, 2003, **40**, 970–979.

[5] Zeifman A. I. Stability for contionuous-time nonhomogeneous Markov chains. *Lect. Notes Math.*, 1985, **1155**, 401–414.

[6] Zeifman A. Stability of birth and death processes, *Journal of Mathematical Sciences*, 1998, **91**, 3023-3031.

[7] Zeifman A., Leorato S., Orsingher E., Satin Ya., Shilova G. Some universal limits for nonhomogeneous birth and death processes. *Queueing Syst.*, 2006, **52**, 139–151.

[8] A. I. Zeifman, V. E. Bening, I.A. Sokolov: Continuous-time Markov chains and models (in Russian). Moscow, Elex-KM 2008.

[9] Zeifman A. I. On the nonstationary Erlang loss model, *Automation and Remote Control*, 2009, **70**, 2003–2012.