On stability for $M_t/M_t/N/N$ queue

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Abstract—We obtain some stability bounds for nonstationary Erlang loss queueing model. Two specific examples of queues with close to periodic arrival and service rates are considered.

Index Terms—queueing system; stability bounds; nonstationary Erlang model

I. INTRODUCTION

In this note we consider the estimates of stability for the simplest nonstationary Erlang queueing model. There is a number of investigations of nonstationary continuous-time Markov chains, see for instance first results in [5], and more detail studies for birth and death processes (BDPs) in [1], [6]. Now we consider nonstationary $M_t/M_t/N/N$ queue and obtain some new and simple stability bounds.

Let $X = X(t)$, $t \geq 0$ be queue-length process for $M_t/M_t/N/N$ queue. This is a BDP on state space $E_N = \{0, 1, \ldots, N\}$ and birth and death rates $\lambda_i(t) = \lambda(t), \mu_i(t) = n_i(t)$ respectively. We suppose that arrival and service intensities $\lambda(t)$ and $\mu(t)$ are locally integrable on $[0, \infty)$. Let $p_i(t) = Pr\{X(t) = i\}$ be state probabilities of $X(t)$, and $p(t) = (p_0(t), \ldots, p_N(t))^T$ be the respective column vector.

Then we have the forward Kolmogorov system

$$
\frac{dp}{dt} = \begin{cases} 
\lambda(t)p_{k-1} - (\lambda(t) + k\mu(t))p_k + (k+1)\mu(t)p_{k+1}, & 1 \leq k < N, \\
\lambda(t)p_{N-1} - N\mu(t)p_N & \text{otherwise}
\end{cases}
$$

in the following form:

$$
\frac{dp}{dt} = A(t)p, \quad t \geq 0,
$$

where $A(t) = \{a_{ij}(t), t \geq 0\}$ is the transposed intensity matrix of the process, and

$$
a_{ij}(t) = \begin{cases} 
\lambda(t), & \text{if } j = i - 1, \\
(i+1)\mu(t), & \text{if } j = i + 1, \\
(\lambda(t) + i\mu(t)), & \text{if } j = i, \\
0, & \text{otherwise}.
\end{cases}
$$

We denote throughout the paper by $\| \cdot \|$ the $l_1$-norm, i.e. $\|x\| = \sum |x_i|$, for $x = (x_0, \ldots, x_N)^T$ and $\|B\| = \max_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}$. Let $\Omega = \{x : x \geq 0, \|x\| = 1\}$ be a set of all stochastic vectors.

Let $E_k(t) = E\{X(t) | X(0) = k\}$ be the mean of the process at the moment $t$ under initial condition $X(0) = k$, and $E_p(t)$ be the mathematical expectation (the mean) at the moment $t$ under initial probability distribution $p(0) = p$.

Consider also a "perturbed" queue-length process $\hat{X} = \hat{X}(t), t \geq 0$ with general structure of intensity matrix $\hat{A}(t)$. In general, $\hat{X}(t)$ is not BDP. Put $\hat{A}(t) = \hat{A}(t) - A(t)$. We assume that the perturbations are uniformly small, i.e. $\|\hat{A}(t)\| \leq \varepsilon$ for almost all $t \geq 0$.

II. STABILITY BOUNDS

Let $d_1, \ldots, d_N$ be positive numbers. Consider the following expression:

$$
\alpha_i(t) = \lambda(t) + i\mu(t) - \frac{d_{i+1}}{d_i}\lambda(t) - \frac{d_i}{d_i-1}(i-1)\mu(t),
$$

for $i = 1, 2, \ldots, N$, $t \geq 0$. Where $d_0 = d_{N+1} = 0$. Put $G = \sum_{i=1}^N d_i$ and $d = \min_i d_i$.

Theorem 1. Let there exist a positive sequence $\{d_i\}$ and a positive number $\beta$ such that

$$
\alpha_i(t) \geq \beta, \quad i = 1, 2, \ldots, N, \quad t \geq 0.
$$

Then the following stability bounds hold:

$$
\limsup_{t \to \infty} \|p(t) - \bar{p}(t)\| \leq \frac{\varepsilon(1 + \log 4G)}{\beta},
$$

and

$$
\limsup_{t \to \infty} |E_p(t) - \bar{E}_p(t)| \leq \frac{N\varepsilon(1 + \log 4G)}{\beta},
$$

for arbitrary initial probability distributions $p(0)$ and $\bar{p}(0)$ for $X(t)$ and $\hat{X}(t)$ respectively.

Proof. Firstly we find the basic estimate of the rate of convergence. The property $\sum_{i=0}^N p_i(t) = 1$ for any $t \geq s$ allows to put $p_0(t) = 1 - \sum_{i=1}^N p_i(t)$, then we obtain the following system from (2)

$$
\frac{dz(t)}{dt} = B(t)z(t) + f(t),
$$
where \( z(t) = (p_1(t), \ldots, p_N(t))^T \), \( f(t) = (\lambda_0(t), 0, \ldots, 0)^T \), \( B(t) = (b_{ij}(t))_{i,j=1}^N \) and respective \( b_{ij}(t) \), see details in [7], [8]. Consider now the triangular matrix
\[
D = \begin{pmatrix}
  d_1 & d_1 & \cdots & d_1 \\
  0 & d_2 & \cdots & d_2 \\
  0 & 0 & \ddots & \ddots \\
  0 & 0 & 0 & d_N
\end{pmatrix},
\]
and the respective norms \( \|x\|_{1D} = \|Dx\| \), and \( \|B\|_{1D} = \|DBD^{-1}\| \).

We have now the following bound of the logarithmic norm \( \gamma(B(t)) \) in \( 1D\)-norm (see for instance [2], [3], [7], [9]):
\[
\gamma(B)_{1D} = \max_{i \geq 0} \left( \frac{d_{i+1}}{d_i} \lambda(t) + \frac{d_{i-1}}{d_i} \mu(t) - (\lambda(t) + (i+1)\mu(t)) \right) = \max_{i \geq 0} \left( -\alpha_i(t) \right) \leq -\beta,
\]
in accordance with (5). Therefore the following inequality holds:
\[
\|z^*(t) - z^{**}(t)\|_{1D} \leq e^{-\beta(t-s)}\|z^*(s) - z^{**}(s)\|_{1D},
\]
for any initial conditions \( z^*(s) \), \( z^{**}(s) \) and any \( s, t, 0 \leq s \leq t \). Then we obtain
\[
\|p^*(t) - p^{**}(t)\| \leq 2\|z^*(t) - z^{**}(t)\| = 2\|D^{-1}D (z^*(t) - z^{**}(t))\| \leq 4\|D^{-1}\| \|z^*(t) - z^{**}(t)\|_{1D} \leq 4\|D^{-1}\|\frac{d}{d} e^{-\beta(t-s)}\|z^*(s) - z^{**}(s)\|_{1D} \leq 4\|D^{-1}\|\frac{d}{d} e^{-\beta(t-s)}\|p^*(s) - p^{**}(s)\| \leq \frac{8G}{d} e^{-\beta(t-s)},
\]
for any initial conditions \( p^*(s) \), \( p^{**}(s) \) and any \( s, t, 0 \leq s \leq t \).

Consider the forward Kolmogorov system for perturbed process:
\[
\frac{dp}{dt} = \tilde{A}(t)p(t).
\]

We can apply the approach of paper [4]. Put
\[
\beta(t, s) = \sup_{\|v\| = 1, \sum v_i = 0} \left\| U(t, s) v \right\| = \frac{1}{2} \max_{i,j} \sum_k |p_{ik}(t, s) - p_{jk}(t, s)|,
\]
where \( U(t, s) \) is Cauchy matrix of (2), and \( p_{ik}(t, s) = Pr \{X(t) = k|X(s) = i\} \). Mitrophanov in [4] proved the bound of stability, that in the nonstationary case is the following one:
\[
\|p(t) - \bar{p}(t)\| \leq \beta(t, s)\|p(s) - \bar{p}(s)\| + \int_s^t \|\tilde{A}(u)\|\beta(u, s)\, du.
\]

Moreover, the following estimates hold:
\[
\beta(t, s) \leq 1, \quad \beta(t, s) \leq \frac{ce^{-b(t-s)}}{2}, \quad 0 \leq s \leq t,
\]
where under our assumptions \( c = \frac{8G}{d} \), \( b = \beta \). Finally the following stability bound holds:
\[
\|p(t) - \bar{p}(t)\| \leq \left\{ \begin{array}{ll}
\|p(s) - \bar{p}(s)\| + (t-s)\varepsilon, & 0 < t - s < b^{-1} \log \frac{c}{2}, \\
\frac{b^{-1}(\log \frac{c}{2} + 1 - ce^{-b(t-s)})\varepsilon + \frac{1}{2}e^{-b(t-s)}\|p(s) - \bar{p}(s)\|, & t - s \geq b^{-1} \log \frac{c}{2}, \end{array} \right.
\]
for any initial conditions \( p(s) \), \( \bar{p}(s) \). Let \( t - s \to \infty \). Then (17) implies our claim.

Let now non-perturbed process has 1-periodic intensities. Then we can obtain the following stability result.

**Theorem 2.** Let \( \lambda(t) \) and \( \mu(t) \) be 1-periodic. Let there exist a positive sequence \( \{d_i\} \) and a positive number \( \beta^* \) such that
\[
\alpha_i(t) \geq \beta(t), \quad i = 1, 2, \ldots, N, 0 \leq t \leq 1,
\]
where
\[
\int_0^1 \beta(t)\, dt \geq \beta^*.
\]

Let
\[
K = \sup_{|t-s| \leq 1} \int_s^t \beta(\tau)\, d\tau < \infty.
\]

Then the following stability bounds hold:
\[
\limsup_{t \to \infty} \|p(t) - \bar{p}(t)\| \leq \frac{\varepsilon \left( 1 + \log \frac{4Ge^K}{d} \right)}{\beta^*},
\]
and
\[
\limsup_{t \to \infty} |E_p(t) - \bar{E}_p(t)| \leq \frac{N\varepsilon \left( 1 + \log \frac{4Ge^K}{d} \right)}{\beta^*},
\]
for arbitrary initial probability distributions \( p(0) \) and \( \bar{p}(0) \) for \( X(t) \) and \( \bar{X}(t) \) respectively.

**Proof.** We have
\[
e^{-\int_s^t \beta(u)\, du} \leq Re^{-\beta^*(t-s)},
\]
where \( R = e^K \), and now \( c = \frac{8Ge^K}{d} \), \( b = \beta^* \).
III. Examples

Example 1.
Let \( \lambda(t) = 2 + \sin 2\pi t, \mu(t) = 1 + \cos 2\pi t, \varepsilon = 10^{-6} \). Put all \( d_i = 1 \). Then we have

\[
\beta(t) = \mu(t) = 1 + \cos 2\pi t, \quad G = N, \quad d = 1.
\] (24)

Therefore, instead of (6) and (7) we can write the following estimates

\[
\|p^*(t) - p^{**}(t)\| \leq 8Ne^{-\int_0^t (1+\cos 2\pi u) \, du} \leq 16Ne^{-t}, \tag{25}
\]

and

\[
\|E_{p^*} - E_{p^{**}}\| \leq 16N^2e^{-t}. \tag{26}
\]

Let \( N = 5000 \). We have the following stability bounds:

\[
lm sup \|p(t) - \bar{p}(t)\| \leq 1.3 \cdot 10^{-5}
\]

and

\[
lm sup |E_{p(t)} - E_{\bar{p}(t)}| \leq 0.065.
\]

Now, \( 16N e^{-t} \leq 2.035 \cdot 10^{-8} \) and \( 16N^2 e^{-t} \leq 1.02 \cdot 10^{-4} \) for \( t \geq 29 \), and we can find the limiting characteristics of original and perturbed processes approximately at interval \( t \in [29, 30] \). Namely, we have limiting loss probability \( \bar{p}_N(t) \approx p_N(t) \approx 0 \), double mean \( \bar{E} \approx E = \lim_{t \to \infty} \frac{\int_0^t \phi(u) \, du}{t} \approx 2.103 \), limiting probability of empty queue \( \bar{p}_0(t) \approx p_0(t) \) is shown in Fig.1, and the limiting mean \( \phi(t) \approx \bar{\phi}(t) \) is shown in Fig.2.

Example 2.
Let now \( \lambda(t) = N(2 + \sin 2\pi t), \mu(t) = 1 + \cos 2\pi t, \varepsilon = 10^{-6} \). Put \( d_i = 1 \). Then we have

\[
\beta(t) = 1 + \cos 2\pi t, \quad G = N, \quad d = 1. \tag{27}
\]

Therefore, instead of (6) and (7) we can write the same bounds

\[
\|p^*(t) - p^{**}(t)\| \leq 8Ne^{-\int_0^t (1+\cos 2\pi u) \, du} \leq 16Ne^{-t}, \tag{28}
\]

and

\[
\|E_{p^*} - E_{p^{**}}\| \leq 16N^2e^{-t}. \tag{29}
\]

Let \( N = 1000 \). We have the following stability bounds:

\[
lm sup \|p(t) - \bar{p}(t)\| \leq 1.14 \cdot 10^{-5},
\]

and

\[
lm sup |E_{p(t)} - E_{\bar{p}(t)}| \leq 0.012.
\]

Now, \( 16N e^{-t} \leq 4.07 \cdot 10^{-9} \), \( 16N^2 e^{-t} \leq 4.0699 \cdot 10^{-6} \) for \( t \geq 29 \), and we can find the limiting characteristics of original and perturbed processes approximately at interval \( t \in [29, 30] \).

Namely, we have limiting probability of empty queue \( \bar{p}_0(t) \approx p_0(t) \approx 0 \), double mean \( \bar{E} \approx E = \lim_{t \to \infty} \frac{\int_0^t \phi(u) \, du}{t} \approx 984.64 \), limiting loss probability \( \bar{p}_N(t) \approx p_N(t) \) is shown in Fig.3, and the limiting mean \( \phi(t) \approx \bar{\phi}(t) \) is shown in Fig.4.
Fig. 1: Example 1. Approximation of the limit behavior of $\tilde{p}_0(t) = \Pr (\tilde{X}(t) = 0)$.

Fig. 2: Example 1. Approximation of the limiting mean $\tilde{\phi}(t) = \tilde{E}_0(t)$.

Fig. 3: Example 2. Approximation of the limit behavior of loss probability $\tilde{p}_N(t)$.

Fig. 4: Example 2. Approximation of the limiting mean $\tilde{\phi}(t) = \tilde{E}_0(t)$.

REFERENCES